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A Renormalization Group Analysis of Correlation Functions for the Dipole Gas

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We develop a renormalization group method for analyzing the generating functional for charge correlations of a dilute classical dipole gas. It is based on and extends the renormalization group analysis introduced by Brydges and Yau for the dipole gas partition function. Our method leads to systematic formulas for the large-distance behavior of correlation functions of all orders. We prove that in any dimension $d \geqslant 2$, at any value $\beta > 0$ of the inverse temperature, and at sufficiently small activity z, the correlation functions exhibit at large distances the same behavior as for a vacuum ($z = 0$), but with a new dielectric constant $1 + \sigma$ over which we have good control. The results proved here extend existing results on the two-point correlations to all higher correlations, and constitute a general confirmation of the fact that dipoles do not screen.

KEY WORDS: Dipole gas; renormalization group; correlation functions; large-distance asymptotics.

1. INTRODUCTION

The potential between two unit dipoles $p, p' \in S^{d-1}$ located at $x, y \in \mathbf{R}^d$ has the form

$$(p \cdot \partial)(p' \cdot \partial)(-\Delta)^{-1}(x - y) \tag{1}$$

where the kernel of the inverse Laplacian $(-\Delta)^{-1}$ is the Coulomb potential. The classical statistical mechanics of a gas of such dipoles with

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temperature β^{-1} and fugacity z is given by the grand canonical partition function

$$Z=\sum_{n\geqslant 0}\int_{(S^{d-1}\times\mathbf{R}^d)^n}\prod_i dp_i\,dx_i\,(z^n/n!)$$
$$\times\exp(-(\beta/2)\sum_{i,j}(p_i\cdot\partial)(p_j\cdot\partial)(-\Delta)^{-1}(x_i-x_j)) \tag{2}$$

The model can equivalently be expressed as a Euclidean sine-Gordon quantum field theory by

$$Z=\int e^{-V(\phi)}\,d\mu_v(\phi) \tag{3}$$

Here ϕ is a scalar function on $\mathbf{R}^d$, $d\mu_v$ is a Gaussian measure with covariance $v=\beta(-\Delta)^{-1}$, and

$$V(\phi)=2z\int_{S^{d-1}\times\mathbf{R}^d}dp\,dx\cos[p\cdot\partial\phi(x)] \tag{4}$$

This model has both short-distance and large-distance difficulties. The short-distance problem is not physically important and is removed by introducing a short-distance cutoff on the Coulomb potential. Our concern is to control the large-distance behavior of the theory. We shall analyze the model in finite, arbitrarily large volumes, and prove bounds which are uniform in the volume.

We study the generating functional for charge correlations

$$Z(\rho)=\langle e^{i(\rho,\phi)}\rangle=Z^{-1}\int e^{i(\rho,\phi)}e^{-V(\phi)}\,d\mu_v(\phi) \tag{5}$$

where $(\rho,\phi)=\int\rho(x)\,\phi(x)\,dx$. Then $Z(\rho)$ determines the partition function for a gas with an *a priori* charge distribution $\rho(x)$. Correlations between a charge $+q$ at x and $-q$ at x' are described by the charge correlation functions

$$\begin{aligned} g(x,y)&=Z(q\delta_x-q\delta_y)\\ g^t(x,y)&=Z(q\delta_x-q\delta_y)-Z(q\delta_x)\,Z(-q\delta_y)\end{aligned} \tag{6}$$

We also consider the field correlation functions

$$G(x_1,\dots,x_p)=\langle\phi(x_1)\cdots\phi(x_p)\rangle=i^p[\delta/\delta\rho(x_1)\cdots\delta/\delta\rho(x_p)\,Z(\rho)]\,|_{\rho=0} \tag{7}$$

as well as the truncated functions defined by

$$G^t(x_1,\dots,x_p)=i^p\{\delta/\delta\rho(x_1)\cdots\delta/\delta\rho(x_p)[\log Z(\rho)]\}\,|_{\rho=0} \tag{8}$$

The present paper does not deal with dipole correlation functions (expectations of $\partial\phi$). However, with little extra effort, the method can be extended to treat the generating functional for such correlations.

A detailed analysis of the dipole gas model and $(\partial\phi)^4$ model was given by Gawedzki and Kupiainen[5,6] (see also the similar work of Magnen and Seneor[7]). They implement the ultraviolet cutoff by working on a lattice and show that under iteration of the renormalization group (block-spin) transformation, the effective interactions tend to zero, leaving a free field Gaussian piece μ_{v^*} with $v^* \sim v$. Furthermore, the pressure and dielectric constant are shown to be analytic in the fugacity z. They then extend the result to show that the long-distance behavior of correlation functions (in particular G_2 and G_4^t) is the same as that of the fixed point μ_{v^*}.

More recently, Brydges and Yau[1] introduced new renormalization group techniques for the continuum dipole model described above. They reproduced the first result of Gawedzki and Kupiainen, the convergence of the effective interactions to zero and the analyticity of the pressure. Their method is conceptually simpler and has great potential for generalization to other models. In fact, we have extended this method to analyze the $d=2$ Coulomb gas in the Kosterlitz–Thouless phase[2] and quantum electrodynamics itself.[3]

In outline, the method goes as follows. Beginning with the model on a d-dimensional torus $\Lambda(N)$ of linear size L^N, $L \geqslant 2$, the measure $e^{-V}\,d\mu$ on $\Lambda(N)$ is replaced by a sequence of measures $e^{-V^j}\,d\mu_j$ on smaller tori $\Lambda(N-j)$ for $0 \leqslant j \leqslant N$. The sequence is generated by integrating out short-distance modes and rescaling. The change in the partition function in passing from j to $j+1$ is given by a multiplicative factor Z_j depending on V_j, and one has the representation

$$Z = \left[\prod_{i<j} Z_i\right] \int e^{-V^j(\phi)}\,d\mu_j(\phi) \tag{9}$$

expressing Z as a product of contributions from different length scales. If z is small enough (i.e., the gas is dilute), it is proved that the effective potentials V_j tend to zero as $j \to \infty$. Furthermore, $\log Z_j/|\Lambda|$ tends to zero exponentially fast, uniformly in N, and this controls the pressure $\log Z/|\Lambda|$.

A key feature of the Brydges–Yau approach is that the Gibbs factors e^{-V^j} at each stage are expressed as polymer expansions

$$e^{-V^j(\phi)} = \sum_{\{X_i\}} \prod_i K^j(X_i, \phi) \tag{10}$$

where a polymer X is a union of unit hypercubes ("blocks") in Λ_N. All results follow from estimates showing $K^j(X, \phi) \to 0$ as $j \to \infty$, while maintaining control over growth in ϕ and decay in the size of X.

The goal of the present paper is to extend the work of Brydges and Yau to a complete analysis of large-distance asymptotics of correlations in the dipole gas. We prove formulas similar to (9) for the quantity $Z(\rho)$:

$$Z(\rho)=\left[\prod_{i<j} Z_i(\rho)\right]\left[\frac{\int e^{i(\rho^j,\phi)}e^{-V^j(\phi,\rho)}\,d\mu_j(\phi)}{\int e^{-V^j(\phi)}\,d\mu_j(\phi)}\right] \tag{11}$$

where now $e^{-V^j(\phi,\rho)}$ is expressed as a polymer expansion involving activities $K^j(\rho)=K^j(X,\phi,\rho)$. The important issue is to control the dependence on ρ in addition to the X, ϕ dependence. Our main result (Theorem 1) says that $K^j(\rho)$ is analytic in ρ around $\rho=0$ and that the functional derivatives $[\delta/\delta\rho(x_1)\cdots\delta/\delta\rho(x_p)\,K^j]\,|_{\rho=0}$ have good decay as the points separate. In addition, as j gets large, the $K^j(\rho)$ still tend to zero exponentially fast and the factors $Z_j(\rho)$ tend to appropriate Gaussians. This gives control over the generating functional, and hence a systematic treatment of correlation functions of all orders. We obtain explicit asymptotic formulas for correlation functions which extend the results of Gawedzki and Kupiainen. Theorem 2 bounds the 2-point function:

$$\begin{aligned}&|G_2(x,y)-(1+\sigma)^{-1}\,\beta(-\Delta)^{-1}\,(x,y)|\\ &\qquad\leqslant \mathcal{O}(1)\,|x-y|^{-d+1+\varepsilon},\qquad 0<\varepsilon<1\end{aligned} \tag{12}$$

The factor $1+\sigma$ is a dielectric constant. The truncated correlation functions G_n^t for values $n\geqslant 4$ are bounded by

$$|G^t(x_1,\dots,x_p)|\leqslant \mathcal{O}(1)\,p!\,u_0^{-p}T(x_1,\dots,x_p)^{-d/2+1}\,\mathrm{diam}(x_1,\dots,x_p)^{-d/2+\varepsilon} \tag{13}$$

for some constant u_0. Here $T(x_1,\dots,x_p)$ is the product of the lengths of the bonds for a shortest tree on the points. Theorem 3 gives upper and lower bounds for the charge correlation functions $g(x,y)$, $g^t(x,y)$. It also shows the vanishing of $Z(q\delta_x)=\langle e^{iq\phi(x)}\rangle$ in two dimensions as the volume gets large. These last results are analogous to bounds proved by Fröhlich and Spencer[4] for a class of lattice dipole gases.

We make a final remark concerning the thermodynamic limit of the Brydges–Yau procedure. This is to note that there is a well-defined RG transformation $K^j\to K^{j+1}$ and generating functional $Z(\rho)$ for the infinite volume $\mathbf{R}^d$. This we can see because Eqs. (16), (24), (26), (27), (31), (32), (34), and (43) [plus (29) if suitably reexpressed] make perfect sense for infinite volume. All of our results, and those of ref. 1, if appropriately interpreted, hold in this setting. This, however, does not prove the existence of the limit $\{K\}_{\Lambda(N)}\to\{K\}_{\mathbf{R}^d}$. We have not yet shown that the infinite-volume RG transformation is achieved as the $N\to\infty$ limit of RG transformations on tori $\Lambda(N)$.

2. RENORMALIZATION GROUP TRANSFORMATIONS

We begin with the precise definition of the model. The modified inverse Laplacian on the d-dimensional torus $\Lambda = \Lambda(N)$ is the operator v with the smooth kernel

$$v(x, y) = |\Lambda|^{-1} \sum_{p \in \Lambda^*} e^{ip(x-y)} \tilde{v}(p)$$
$$\tilde{v}(p) = \beta p^{-2} e^{-p^4}(1 - \delta_{p,0}) \tag{14}$$

where $\Lambda^* = (2\pi L^{-N}\mathbf{Z})^d$. Then v is positive definite on functions f on Λ with $\int f = 0$ and so defines a measure μ_v on a suitable Sobolev space $\mathscr{H}_s(\Lambda)$ of (continuous) functions of this type. The initial interaction e^{-V} given by (4) is rewritten in a polymer expansion

$$e^{-V(\phi)} = \sum_{\{X_i\}} \prod_i K^0(X, \phi) = \mathscr{E}xp[\square + K^0(\phi)] \tag{15}$$

The activities K^0 are given by

$$K^0(X, \phi) = \begin{cases} \prod_{\Delta \in X} [\exp(-V_\Delta) - 1] & \text{if } X \text{ is connected} \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

where $V_\Delta(\phi)$ is the potential integrated over a unit block Δ. The sum is over disjoint polymers, where a polymer X is a union of closed unit blocks Δ with centers on $\mathbf{Z}^d \cap \Lambda$. Following ref. 1, Section 1, we have used the "circle exponential" representation for the polymer expansion. The generating functional (5) may now be written

$$Z(\rho) = \langle e^{i(\rho,\phi)} \rangle = \int e^{i(\rho,\phi)} \mathscr{E}xp[\square + K^0(\phi)]\, d\mu_{v^0}(\phi)/[\rho = 0] \tag{17}$$

We allow ρ to be a measure of bounded variation on Λ and interpret (ρ, ϕ) as $\int \phi(x)\, d\rho(x)$.

Our goal is to perform transformations on this integral leading to similar functionals

$$\langle e^{i(\rho,\phi)} \rangle_{\rho,j} = \int e^{i(\rho,\phi)} \mathscr{E}xp(\square + K^j)(\phi, \rho)\, d\mu_{v^j}(\phi)/[\rho = 0] \tag{18}$$

where ϕ, ρ, etc., are now defined on $\Lambda_j \equiv \Lambda(N-j)$. Here the covariance has Fourier transform

$$\tilde{v}^j(p) = \beta p^{-2}(e^{p^4} + \sigma^j)^{-1}(1 - \delta_{p,0}) \tag{19}$$

where $1+\sigma^j$ turns out to be an effective dielectric constant. The functionals $K^j(X, \phi, \rho)$ are assumed to come from functionals $K^j(X, \psi, \rho)$ defined on pairs $\psi = (\psi_\mu, \psi_{\mu\nu})$, $\mu, \nu = 1,..., d$, by restricting to $\psi = \psi_\phi = (\partial_\mu \phi, \partial_{\mu\nu}\phi)$. The $K^j(X, \psi, \rho)$ are to be analytic in ψ on a neighborhood of the subspace $\psi = \psi_\phi$ and analytic in ρ around $\rho = 0$ (we are more precise about this in Section 3).

We now define a single renormalization group transformation from j to $j+1$. In the special case $\rho = 0$ the analysis reduces to that of ref. 1. The first step is a fluctuation integral integrating out high frequencies. We define, for integer $L \geqslant 2$, an operator $v^{\#} = v^{\#,j}$ by

$$\tilde{v}^{\#}(p) = \beta p^{-2}(e^{L^4 p^4} + \sigma^j)^{-1}(1 - \delta_{p,0}) \tag{20}$$

and define a new covariance operator

$$C^j = v^j - v^{\#} \tag{21}$$

Then we have

$$\langle e^{i(\rho,\phi)} \rangle_{\rho,j} = \int e^{i(\rho,\phi)}[\cdots]\, d\mu_{v^{\#}}(\phi)/[\rho = 0] \tag{22}$$

where

$$\begin{aligned}[\cdots] &\equiv \int e^{i(\rho,\zeta)}\, \mathscr{E}xp(\square + K^j)(\zeta + \phi, \rho)\, d\mu_{C^j}(\zeta) \\ &= \exp[-\tfrac{1}{2}(\rho, C^j\rho)] \int \mathscr{E}xp(\square + K^j)(\zeta + \phi + iC^j\rho, \rho)\, d\mu_{C^j}(\zeta) \\ &= \exp[-\tfrac{1}{2}(\rho, C^j\rho)]\, \mathscr{E}xp(\square + K^{\#}(\phi, \rho)) \end{aligned} \tag{23}$$

Here we take advantage of the analyticity in ϕ to make a complex contour translation $\zeta \to \zeta + iC^j\rho$. The functional $K^{\#}$ is obtained from K^j in two steps $K^{\#} = \mathscr{T}\mathscr{F}K^j$, where $K \to \mathscr{T}K$ denotes the complex translation step defined on $K(X, \psi, \rho)$ so that

$$(\mathscr{T}K)(X, \phi, \rho) = K(X, \phi + iC^j\rho, \rho) \tag{24}$$

and $K \to \mathscr{F}K$ denotes the fluctuation step defined so that

$$\mathscr{E}xp(\square + \mathscr{F}K) = \mu_{C^j} * [\mathscr{E}xp(\square + K)] \tag{25}$$

Indeed we define $\mathscr{F}K = K(1)$, where $K(t)$ is the solution of

$$K(t) = \mu_{tC^j} * K + \tfrac{1}{2}\int_0^t ds\, \mu_{(t-s)C^j} * (K_\psi(s) \circ K_\psi(s))(C^j) \tag{26}$$

where $K_\psi \circ K_\psi$ is the circle product of functional derivatives.[1]

Next we extract relevant and marginal pieces from $K^{\#}(X, \psi, \rho)$. The point is to isolate and deal with separately the pieces of $K^{\#}$ which do not decrease in size under the RG transformation. From $K^{\#}(X, \psi, 0)$ we extract as in Brydges and Yau. A set X is called small ($X \in \mathscr{S}$) if X is connected and $|X| =$ (the number of unit blocks in X) satisfies $|X| \leqslant 2^d$; as we discuss in Section 3, this definition is a little different from ref. 1. Then define

$$F_0(X, \psi) = K^{\#}(X, 0, 0) - \tfrac{1}{2}\beta^{-1} \sum_{\mu, \nu} F_X \psi_\mu(x)\, \delta\sigma^j_{\mu,\nu}(X)\, \psi_\nu(x)\, dx \tag{27}$$

where

$$\delta\sigma^j_{\mu,\nu}(X) = -\beta/|X| \int [\partial^2 K^{\#}/\partial\psi_\mu(x)\, \partial\psi_\nu(y)](X, 0, 0)\, dx\, dy \tag{28}$$

The extraction is

$$\sum_{X \in \mathscr{S}} F_0(X, \psi) \equiv |\Lambda_j|\, E_j - \tfrac{1}{2}\beta^{-1}\, \delta\sigma^j \int |\psi_\mu|^2 \tag{29}$$

We extract ρ-dependent parts only from $K^{\#}(X, 0, \rho)$. The reason for the ψ-independent extraction here (i.e., no ψ^2 terms) is that ρ-dependent parts of K naturally have improved power counting compared to $\rho = 0$ parts. For $\rho \neq 0$, translation invariance is broken, eliminating volume factors L^d in each RG step.

By analyticity in ρ we have

$$K^{\#}(X, 0, \rho) = \sum_{p=0}^{\infty} 1/p! \int [\delta^p K^{\#}/\delta\rho(x_1) \cdots \delta\rho(x_p)](X, 0, 0) \prod_{i=1}^{p} d\rho(x_i) \tag{30}$$

Let $\delta = (\delta_1, \ldots, \delta_p)$, where each δ_j is a semiopen unit cube whose translates are disjoint and cover Λ. Then we extract small set pieces from (30) and define

$$\mathscr{E}^j(X, \rho) = \sum_{p=1}^{\infty} \sum_{\delta} 1(X \cup \delta_1 \cup \cdots \cup \delta_p \in \mathscr{S}) \times 1/p! \int_\delta [\delta^p K^{\#}/\delta\rho(x_1) \cdots \delta\rho(x_p)](X, 0, 0) \prod_{i=1}^{p} d\rho(x_i) \tag{31}$$

and

$$\mathscr{E}^j(\rho) = \sum_X \mathscr{E}^j(X, \rho) \tag{32}$$

The overall extraction is the sum of (29) and (32) and is implemented by finding a polymer activity $\mathscr{E}K^{\#}$ so that

$$\mathscr{E}xp(\square + K^{\#})(\phi, \rho) = \exp\left\{\mathscr{E}^j(\rho) + |A_j|\, E_j - \tfrac{1}{2}\beta^{-1}\,\delta\sigma^j \int [\partial\phi]^2\right\} \times \mathscr{E}xp(\square + \mathscr{E}K^{\#})(\phi, \rho) \tag{33}$$

The full definition of $\mathscr{E}K^{\#}$ is given in Section 5.

In preparation for a rescaling by a factor L, we introduce a reblocking operation. For a polymer X, let $\bar{X}$ be the smallest L-polymer containing X, where an L-polymer is a union of L-blocks. Then we define, for any function K on polymers, a function $\mathscr{B}K$ on L-polymers U by

$$(\mathscr{B}K)(U) = \sum_{\{X_i\}} \prod_i K(X_i) \tag{34}$$

where the sum is over collections of disjoint polymers $\{X_i\}$ such that $\bigcup_i \bar{X}_i = U$ and the overlap graph on $\{\bar{X}_i\}$ is connected. [The overlap graph is all pairs (i, j) in the index set so $\bar{X}_i \cap \bar{X}_j \neq \varnothing$.] By classifying the terms $\{X_i\}$ in $\mathscr{E}xp(\square + K)$ according to the L-polymers they determine by the above rules, we obtain

$$\mathscr{E}xp(\square + K) = \sum_{\{U_j\}} \prod_j (\mathscr{B}K)(U_j) \equiv \mathscr{E}xp^{(1)}(\square + \mathscr{B}K) \tag{35}$$

Then, defining $K^* = \mathscr{B}\mathscr{E}K^{\#}$, we have

$$\mathscr{E}xp(\square + \mathscr{E}K^{\#}) = \mathscr{E}xp^{(1)}(\square + K^*) \tag{36}$$

Now combining these steps, we get

$$\begin{aligned}\langle e^{i(\rho,\phi)}\rangle_{\rho,j} &= \exp[-\tfrac{1}{2}(\rho, C^j\rho)] \\ &\quad \times \left[\int d\mu_{v^{\#}}\, e^{i(\rho,\phi)} \exp\left\{\mathscr{E}^j(\rho) + |A_j|\, E_j - \tfrac{1}{2}\beta^{-1}\,\delta\sigma^j \int [\partial\phi]^2\right\} \right. \\ &\quad \left. \times\, \mathscr{E}xp(\square + \mathscr{E}K^{\#})\right] \Big/ [\rho = 0] \\ &= \exp[-\tfrac{1}{2}(\rho, C^j\rho) + \mathscr{E}^j(\rho)] \\ &\quad \times \int e^{i(\rho,\phi)}\, \mathscr{E}xp^{(1)}(\square + K^*)\, d\mu_{v^*}/[\rho = 0]\end{aligned} \tag{37}$$

where in the last step we have absorbed $-\frac{1}{2}\beta^{-1}\,\delta\sigma^j \int [\partial\phi]^2$ into $d\mu_{v^\#}$ and defined

$$\begin{aligned}
\tilde{v}^*(p) &= \beta p^{-2}(e^{L^4 p^4} + \sigma^{j+1})^{-1}\,(1-\delta_{p,0}) \\
\sigma^{j+1} &= \sigma^j + \delta\sigma^j
\end{aligned} \tag{38}$$

Finally, we scale by a factor L to obtain a theory on Λ_{j+1}. We set $K^{j+1} = \mathscr{S}K^*$, where the scaling operator $\mathscr{S}$ is defined on $K(X, \psi, \rho)$, so that

$$(\mathscr{S}K)(X, \phi_L, \rho_L) = K(LX, \phi, \rho) \tag{39}$$

with $\phi_L(x) = L^{d/2-1}\phi(Lx)$, $\rho_L(x) = L^{d/2+1}\rho(Lx)$. Then we have

$$\langle e^{i(\rho,\phi)}\rangle_{\rho,j} = \exp[-\tfrac{1}{2}(\rho, C^j\rho) + \mathscr{E}^j(\rho)]\,\langle e^{i(\rho_L,\phi)}\rangle_{\rho_L,j+1} \tag{40}$$

Now we iterate this. Given ρ on Λ_0, define ρ^j on Λ_j for $0 \leqslant j \leqslant N$ by $\rho^j(x) = \rho_{L^j}(x)$, and then

$$Z(\rho) = \exp\left\{\sum_{j=0}^{N-1} [-\tfrac{1}{2}(\rho^j, C^j\rho^j) + \mathscr{E}^j(\rho^j)]\right\}\langle e^{i(\rho^N,\phi)}\rangle_{\rho^N,N} \tag{41}$$

This is the expansion. Note that the exponential has the form $\prod_{j=0}^{N-1} Z_j(\rho)$ referred to in the introduction.

It is convenient to take one last complex translation and fluctuation step. With the special definitions $C^N \equiv v^N$, $\Lambda_N = \Delta$ (a unit block), this yields

$$\begin{aligned}
\langle e^{i(\rho^N,\phi)}\rangle_{\rho^N,N} &= \exp[-\tfrac{1}{2}(\rho^N, C^N\rho^N)]\,\mathscr{E}xp(\square + K^{\#,N})(0, \rho^N)/[\rho=0] \\
&= \exp[-\tfrac{1}{2}(\rho^N, C^N\rho^N)](1 + K^{\#,N})(\Delta, 0, \rho^N)/[\rho=0] \\
&\equiv \exp[-\tfrac{1}{2}(\rho^N, C^N\rho^N) + \mathscr{E}^N(\rho^N)]
\end{aligned} \tag{42}$$

Here we have made the special definition $\mathscr{E}^N(\rho) = \log\{(1+K^\#)(\Delta, 0, \rho)/[\rho=0]\}$. Thus, we find the formula

$$Z(\rho) = \exp[-\tfrac{1}{2}(\rho, w\rho) + \mathscr{E}(\rho)] \tag{43}$$

where

$$\begin{aligned}
(\rho, w\rho) &= \sum_{j=0}^{N} (\rho^j, C^j\rho^j) \\
\mathscr{E}(\rho) &= \sum_{j=0}^{N} \mathscr{E}^j(\rho^j)
\end{aligned} \tag{44}$$

3. NORMS

We control the development of the polymer activities $K(X, \psi, \rho)$ by estimating the size of certain norms which measure growth in ψ, decay in X, and analyticity in ψ and ρ.

The variables $\psi = (\psi_1, \psi_2)$ and ρ take values in Banach spaces which we now specify. For $\psi_1 = (\psi_\mu)$ and $\psi_2 = (\psi_{\mu\nu})$ we assume that the components are elements of $C(\Lambda)$, the continuous functions on Λ with sup norm, equivalently, $\psi_\alpha \in C(\Lambda \times \Omega_\alpha)$, where Ω_α is an index space, $\alpha = 1, 2$. For the measure ρ we assume it is an element of the dual space $C'(\Lambda)$, which is identified as the regular Borel measures with the total variation norm.

We assume that $K(X, \psi, \rho)$ is analytic in ψ on an open strip around the real subspace $\psi = \psi_\phi = (\partial_\mu \phi, \partial_\mu \partial_\nu \phi)$, $\phi \in \mathscr{H}_s(\Lambda)$, and analytic in ρ in a ball around 0.

For $n = (n_1, n_2)$ and p let $K_{n,p}(X, \phi)$ be the derivative of $K(X, \psi, \rho)$ of order n with respect to ψ at $\psi = \psi_\phi$ and of order p with respect to ρ at $\rho = 0$. This is a continuous multilinear functional on $C(\Lambda \times \Omega_1)^{n_1} \times C(\Lambda \times \Omega_2)^{n_2} \times C'(\Lambda)^p$, symmetric in each of the three entries. We further assume that this functional is given by integration against a bounded Borel function from Λ^p to $C'(\hat{\Lambda}^n)$, where $\hat{\Lambda}^n = (\Lambda \times \Omega_1)^{n_1} \times (\Lambda \times \Omega_2)^{n_2}$. The value at $\mathbf{x} = (x_1, ..., x_p)$ is a measure on $\hat{\Lambda}^n$ which we denote $K_{n,p}(X, \phi, \mathbf{x})$. Formally, we might represent the measure by a function $K_{n,p}(X, \phi, \xi, \mathbf{x})$ with $\xi = (\xi_1^1, ..., \xi_{n_1}^1, \xi_1^2, ..., \xi_{n_2}^2)$ in $\hat{\Lambda}^n$. Then, formally,

$$K_{n,p}(X, \phi, \xi, \mathbf{x}) = \left[\frac{\delta^{n+p} K(X, \psi, \rho)}{\delta\rho(x_1) \cdots \delta\rho(x_p)\, \delta\psi(\xi_1^1) \cdots \delta\psi(\xi_{n_2}^2)} \right] \Bigg|_{\psi = \psi_\phi, \rho = 0} \tag{45}$$

This is the notation we have used in Section 2 [if $\xi = (x, \mu)$, then $\psi(\xi) = \psi_\mu(x)$ and if $\xi = (x, \mu, \nu)$, then $\psi(\xi) = \psi_{\mu\nu}(x)$].

Our basic locality assumption is that the measure $K_{n,p}(X, \phi, \mathbf{x}, \xi)$ has support in ξ in $\hat{X}^n = (X \times \Omega_1)^{n_1} \times (X \times \Omega_2)^{n_2}$ (there are no collars around X as there are in ref. 1). The localization in $\mathbf{x}$ is not strict.

We first define the norm of $K_{n,p}(X, \phi, \mathbf{x})$ to be the total variation norm

$$\|K_{n,p}(X, \phi, \mathbf{x})\| = \sup_{\|F\| < 1} |K_{n,p}(X, \phi, \mathbf{x}; F)| \tag{46}$$

where $F \in C(\hat{X}^n)$. Actually, we usually consider the restriction of the measure to $\Delta = \Delta_{1,1} \times \cdots \Delta_{1,n_1} \times \Delta_{2,1} \times \cdots \Delta_{2,n_2}$ with $\Delta_{ij} \in X$ and so consider $\|K_{n,p}(X, \phi, \mathbf{x})\, 1_\Delta\|$.

Dependence on the variable ϕ is dominated by a large-field regulator $G = G(X, \phi)$ which will have the form

$$G_\kappa(X, \phi) = \exp(\kappa[\|\partial\phi\|_{s,X}^2 + 1/c\ \|\partial\phi\|_{\partial X}^2]), \qquad \kappa > 0 \tag{47}$$

where $\|\partial\phi\|^2_{s,X}$ is the Sobolev norm of order s on X, and c is a constant. This is the choice originally introduced by Gawedzki and Kupiainen.[5] It differs from the $G(X, \phi)$ of Brydges and Yau, who introduce collars around the region X instead of a boundary term. In the Appendix, we show that properties required by ref. 1 are still satisfied by (47). We define

$$\|K_{n,p}(X, \mathbf{x})\|_G = \sup_{\phi \in \mathscr{H}_S(X)} \|K_{n,p}(X, \phi, \mathbf{x})\| \; G^{-1}(X, \phi) \tag{48}$$

Dependence on the set X and points $\mathbf{x}$ is controlled by a large set regulator $\Gamma = \Gamma(X, \mathbf{x})$ to be specified below. We define

$$\|K_{n,p}\|_{G,\Gamma} = \begin{cases} \sup_{x} \sum_{X,\Delta} \Gamma(X, \mathbf{x}) \, \|K_{n,p}(X, \mathbf{x}) \, 1_\Delta\|_G, & p \neq 0 \\ \sup_{\Delta_0} \sum_{\Delta, X \ni \Delta_0} \Gamma(X) \, \|K_{n,0}(X) \, 1_\Delta\|_G & p = 0 \end{cases} \tag{49}$$

We assume translation invariance of K, so that the $p = 0$ norm does not depend on the explicit pin at Δ_0. It is an important feature that for $p \neq 0$, the sum on X is not explicitly pinned (but is implicitly pinned by $\mathbf{x}$). This eliminates for $p > 0$ the volume growth factor L^d which always occurs for $p = 0$ (see Proposition 5).

Finally, for $h = (h_1, h_2)$, $h^n = h_1^{n_1} h_2^{n_2}$ we define

$$\|K\|_{G,\Gamma,h,u} = \sum_{n,p} (h^n/n!)(u^p/p!) \, \|K_{n,p}\|_{G,\Gamma} \tag{50}$$

For K independent of ρ the norm is independent of u and reduces to the norm $\|K\|_{G,\Gamma,h}$ of ref. 1.

If $\|K\|_{G,\Gamma,h,u} < \infty$ for some choice of G, Γ, h, and u, we say that K is a local analytic functional. Such a functional has power series expansions in a strip of width h in ψ and a ball of radius u in ρ.

In the remainder of this section we discuss the large set regulator Γ. For $p = 0$ we make a choice as in ref. 1,

$$\Gamma(X) = A^{|X|} \Theta(X)$$

where $A = L^{d+1}$, $|X|$ is the number of unit blocks in X, and

$$\Theta(X) = \inf_{\tau} \prod_{b \in \tau} \theta(|b|) \tag{51}$$

Here τ is a tree composed of bonds b connecting the centers of blocks in X. Throughout this paper, lengths such as $|b|$ are measured in an l^∞ metric. The function θ satisfies $\theta(s) = 1$ for $s = 0, 1$ and

$$\theta(\{s/L\}) \leqslant L^{-d-1} \theta(s), \qquad s \geqslant 2 \tag{52}$$

where $\{x\}$ denotes the smallest integer greater than x. As $s \to \infty$, $\theta(s) \sim s^{d+1}$.

For $\Gamma(X, \mathbf{x})$ and $p > 0$ we require

$$\Gamma(X, \mathbf{x}) \geqslant 1 \tag{53}$$

$$\Gamma(X \cup Y, (\mathbf{x}, \mathbf{y})) \leqslant \Gamma(X, \mathbf{x})\, \Gamma(Y, \mathbf{y})\, \theta(d(X \cup \delta_{\mathbf{x}}, Y \cup \delta_{\mathbf{y}})) \tag{54}$$

A possible choice is $\Gamma(X, \mathbf{x}) = \Gamma(X \cup \delta_{\mathbf{x}})$, where $\delta_{\mathbf{x}} = \delta_{x_1} \cup \cdots \cup \delta_{x_p}$ and δ_x is a unit square containing x. Then, if $\|K\|_{G,\Gamma,h,u} < \infty$, we have that $K_{n,p}(X, \mathbf{x})$ has power law decrease in the separation of the components of $\mathbf{x}$ from themselves and X.

However, because of the rescaling of ρ at every iteration, we will need to estimate after j steps the decay of $K^j_{n,p}(X, L^{-j}\mathbf{x})$ and still obtain the same power law decay in $\mathbf{x}$. Thus we need extra factors of L^{-1} in the bound for K^j and so extra factors of L in the norm.

Here is the modification we employ. Let $n(\mathbf{x})$ be the number of connected components of $\mathbf{x}$, where we regard two points x_α, x_β as joined if $|x_\alpha - x_\beta| \leqslant a$. Here a is an arbitrary parameter; later we take $a = 2^d$. Define $N_j(\mathbf{x})$ inductively:

$$\begin{aligned} N_0(\mathbf{x}) &= 0 \\ N_{j+1}(L^{-1}\mathbf{x}) &= N_j(\mathbf{x}) + n(\mathbf{x}) \end{aligned} \tag{55}$$

Equivalently, $N_j(\mathbf{x}) = \sum_{k=1}^{j} n(L^k \mathbf{x})$. Then we set

$$\Gamma^j(X, \mathbf{x}) = L^{(d/2-1)\, N_j(\mathbf{x})}\, \Gamma(X \cup \delta_{\mathbf{x}}) \tag{56}$$

Note that (53) and (54) follow by observing that $n(\mathbf{x} \cup \mathbf{y}) \leqslant n(\mathbf{x}) + n(\mathbf{y})$.

To understand the factor $L^{(d/2-1)\, N_j(\mathbf{x})}$, or better, $L^{(d/2-1)\, N_j(L^{-j}\mathbf{x})}$, we have the following result.

Lemma 1. Suppose j is large enough so that $\mathrm{diam}(L^{-j}\mathbf{x}) \leqslant a$. Then

$$L^{(d/2-1)\, N_j(L^{-j}\mathbf{x})} \geqslant L^{(d/2-1)j} (T(\mathbf{x}))^{d/2-1} \tag{57}$$

where

$$T(\mathbf{x}) = \prod_{\substack{(\alpha,\beta) \in \tau \\ |x_\alpha - x_\beta| > a}} (|x_\alpha - x_\beta|/a) \tag{58}$$

Here τ is a tree on $(1,..., p)$ minimizing the total length $\sum_{(\alpha,\beta) \in \tau} |x_\alpha - x_\beta|$.

Proof. The connected components of $\mathbf{x}$ can be obtained by deleting the bonds (α, β) in τ with $|x_\alpha - x_\beta| > a$. Thus, we have

$$n(\mathbf{x}) = 1 + \sum_{(\alpha,\beta) \in \tau} 1(|x_\alpha - x_\beta| > a) \tag{59}$$

The same τ works for $L^{-k}\mathbf{x}$, and so

$$N_j(L^{-j}\mathbf{x}) = \sum_{k=0}^{j-1} n(L^{-k}\mathbf{x}) \tag{60}$$

$$= \sum_{k=0}^{j-1} \left[1 + \sum_{(\alpha,\beta)\in\tau} 1(L^{-k}|x_\alpha - x_\beta| > a) \right]$$

$$\geqslant j + \sum_{\substack{(\alpha,\beta)\in\tau \\ |x_\alpha - x_\beta| > a}} \min(j, \log_L |x_\alpha - x_\beta|/a) \tag{61}$$

Since we assume $j > \log_L |x_\alpha - x_\beta|/a$, the result follows. ∎

On the L-block scale we define

$$\Gamma^{(1),j+1}(U, \mathbf{x}) = L^{(d/2-1)\,N^{(1)}_{j+1}(\mathbf{x})}\Gamma^{(1)}(U \cup \delta^{(1)}_{\mathbf{x}}) \tag{62}$$

where $N^{(1)}_{j+1}(\mathbf{x}) = N_{j+1}(L^{-1}\mathbf{x})$ and $\Gamma^{(1)}(U) = \Gamma(L^{-1}U)$.

In Section 6, we will need to estimate $\Gamma^{(1)}$ in terms of Γ. For $p=0$ we have the result of ref. 1:

$$\Gamma^{(1)}(\bar{X}) \leqslant \begin{cases} \Gamma(X), & X \in \mathscr{S} \\ L^{-d-1}\Gamma(X), & X \notin \mathscr{S} \end{cases} \tag{63}$$

Their proof contains an error, since it uses the implication X *connected* and $X \notin \mathscr{S} \Rightarrow |\bar{X}| < |X|$. There are easy counterexamples. This is why we needed a new definition of small sets. For us the hypotheses imply also $|X| > 2^d$ and then the implication is true, provided $L > 2^d$, which we henceforth assume. A further remark on small sets: An alternative procedure is possible which works for any $L \geqslant 2$. We can take the small sets of ref. 1, but a modified notion of connectedness. This alternative leads to a number of tehnical modifications of ref. 1 which are avoided by our adoption of the clumsier definition of small set and the assumption $L > 2^d$.

For $p \geqslant 1$ we generalize (63) to the following result.

Lemma 2. The following relation holds:

$$\Gamma^{(1),j+1}(\bar{X}, \mathbf{x}) \leqslant \Gamma^{j}(X, \mathbf{x}) \begin{cases} L^{d/2-1}, & X \cup \delta_{\mathbf{x}} \in \mathscr{S} \\ L^{-3}, & X \cup \delta_{\mathbf{x}} \notin \mathscr{S} \end{cases} \tag{64}$$

Remark. This result is not exactly what will be needed in Section 6. For any $\gamma \geqslant 1$, if we choose θ and A so that

$$\theta(\{s/L\}) \leqslant [c(\gamma)\, L^{d+1}]^{-1}\, \theta(s), \qquad A \geqslant c(\gamma)\, L^{d+1} \tag{65}$$

the proof which follows extends to the bounds

$$\Gamma^{(1),j+1}(\bar{X}, \mathbf{x}) \leqslant c(\gamma)\, \gamma^{-|X \cup \delta_{\mathbf{x}}|} \Gamma^j(X, \mathbf{x}) \begin{cases} 1, & p=0, \quad X \in \mathscr{S} \\ L^{-d-1}, & p=0, \quad X \notin \mathscr{S} \\ L^{d/2-1}, & p \geqslant 1, \quad X \cup \delta_{\mathbf{x}} \in \mathscr{S} \\ L^{-3}, & p \geqslant 1, \quad X \cup \delta_{\mathbf{x}} \notin \mathscr{S} \end{cases} \tag{66}$$

The quantity $c(\gamma) = \gamma^{2^d}$ goes to 1 as $\gamma \to 1$. [In ref. 1, $c(\gamma) = 2$.]

Proof. We have

$$\Gamma^{(1),j+1}(\bar{X}, \mathbf{x}) = L^{(d/2-1)\,N_j(\mathbf{x})} L^{(d/2-1)\,n(\mathbf{x})} \Gamma^{(1)}(\bar{X} \cup \bar{\delta}_{\mathbf{x}}) \tag{67}$$

We will show that

$$L^{(d/2-1)\,n(\mathbf{x})} \Gamma^{(1)}(\bar{X} \cup \bar{\delta}_{\mathbf{x}}) \leqslant \begin{cases} L^{d/2-1} \Gamma(X \cup \delta_{\mathbf{x}}), & X \cup \delta_{\mathbf{x}} \in \mathscr{S} \\ L^{-3} \Gamma(X \cup \delta_{\mathbf{x}}), & X \cup \delta_{\mathbf{x}} \notin \mathscr{S} \end{cases} \tag{68}$$

and then the required result follows.

For $n(\mathbf{x}) = 1$, we use (63):

$$L^{(d/2-1)} \Gamma^{(1)}(\bar{X} \cup \bar{\delta}_{\mathbf{x}}) \leqslant \begin{cases} L^{d/2-1} \Gamma(X \cup \delta_{\mathbf{x}}), & X \cup \delta_{\mathbf{x}} \in \mathscr{S} \\ L^{-3} \Gamma(X \cup \delta_{\mathbf{x}}), & X \cup \delta_{\mathbf{x}} \notin \mathscr{S} \end{cases} \tag{69}$$

[Note that $n(\mathbf{x}) = 1$ includes all cases $X \cup \delta_{\mathbf{x}} \in \mathscr{S}$.]

For $n(\mathbf{x}) > 1$ we first suppose $X \cup \delta_{\mathbf{x}}$ is connected. The proof of (68) is by induction on p, the number of points in $\mathbf{x} = (x_1, ..., x_p)$. If $p = 1$, then $n(\mathbf{x}) = 1$ and the result is true. For $p > 1$, let τ be any tree on $X \cup \delta_{\mathbf{x}}$. Since $n(\mathbf{x}) > 1$, there is a pair (x_α, x_β) with $|x_\alpha - x_\beta| > a = 2^d$. The chain of bonds in τ joining δ_{x_α} and δ_{x_β} must then join at least $2^d + 1$ blocks, and any such chain (being a large set) must contain unit blocks δ_1, δ_2 which lie in the same L-block Δ^*. Break the tree between δ_1 and δ_2. Then we have a split $X \cup \delta_{\mathbf{x}} = (X_1 \cup \delta_{\mathbf{x}_1}) \cup (X_2 \cup \delta_{\mathbf{x}_2})$ where each piece is connected and has blocks in Δ^*. Since Δ^* is double counted, we have

$$\Gamma^{(1)}(\bar{X} \cup \bar{\delta}_{\mathbf{x}}) \leqslant L^{-d-1} \Gamma^{(1)}(\bar{X}_1 \cup \bar{\delta}_{\mathbf{x}_1})\, \Gamma^{(1)}(\bar{X}_2 \cup \bar{\delta}_{\mathbf{x}_2}) \tag{70}$$

By the inductive hypothesis we proceed with

$$\begin{aligned} & L^{(d/2-1)\,n(\mathbf{x})} \Gamma^{(1)}(\bar{X} \cup \bar{\delta}_{\mathbf{x}}) \\ & \quad \leqslant L^{(-d-1)} [L^{(d/2-1)\,n(\mathbf{x}_1)} \Gamma^{(1)}(\bar{X}_1 \cup \bar{\delta}_{\mathbf{x}_1})][L^{(d/2-1)\,n(\mathbf{x}_2)} \Gamma^{(1)}(\bar{X}_2 \cup \bar{\delta}_{\mathbf{x}_2})] \\ & \quad \leqslant L^{-3} \Gamma(X_1 \cup \delta_{\mathbf{x}_1})\, \Gamma(X_2 \cup \delta_{\mathbf{x}_2}) \\ & \quad = L^{-3} \Gamma(X \cup \delta_{\mathbf{x}}) \end{aligned} \tag{71}$$

Finally, suppose $X \cup \delta_{\mathbf{x}}$ is not connected. The proof is by induction on the number of connected components of $X \cup \delta_{\mathbf{x}}$. If there is more than one component, we write $X \cup \delta_{\mathbf{x}} = (X_1 \cup \delta_{\mathbf{x}_1}) \cup (X_2 \cup \delta_{\mathbf{x}_2})$, so that $X_1 \cup \delta_{\mathbf{x}_1}$ is connected and

$$\Gamma(X \cup \delta_{\mathbf{x}}) = \Gamma(X_1 \cup \delta_{\mathbf{x}_1})\, \Gamma(X_2 \cup \delta_{\mathbf{x}_2})\, \theta(d(X_1 \cup \delta_{\mathbf{x}_1}, X_2 \cup \delta_{\mathbf{x}_2})) \tag{72}$$

Then we have by (54) and the inductive hypothesis

$$\begin{aligned} &L^{(d/2-1)\,n(\mathbf{x})}\Gamma^{(1)}(\bar{X} \cup \bar{\delta}_{\mathbf{x}}) \\ &\quad \leqslant L^{(d/2-1)\,n(\mathbf{x}_1)}\Gamma^{(1)}(\bar{X}_1 \cup \bar{\delta}_{\mathbf{x}_1})\, L^{(d/2-1)\,n(\mathbf{x}_2)}\Gamma^{(1)}(\bar{X}_2 \cup \bar{\delta}_{\mathbf{x}_2}) \\ &\qquad \times \theta(d^{(1)}(\bar{X}_1 \cup \bar{\delta}_{\mathbf{x}_1}, \bar{X}_2 \cup \bar{\delta}_{\mathbf{x}_2})) \\ &\quad \leqslant L^{-3}\Gamma(X_1 \cup \delta_{\mathbf{x}_1})\, \Gamma(X_2 \cup \delta_{\mathbf{x}_2})\, \theta(d(X_1 \cup \delta_{\mathbf{x}_1}, X_2 \cup \delta_{\mathbf{x}_2})) \\ &\quad = L^{-3}\Gamma(X \cup \delta_{\mathbf{x}}) \end{aligned} \tag{73}$$

Here we used the θ bound (52), which holds since $d^{(1)}(\bar{X}, \bar{Y}) \leqslant L^{-1}d(X, Y)$ and $d(X_1 \cup \delta_{\mathbf{x}_1}, X_2 \cup \delta_{\mathbf{x}_2}) \geqslant 2$. ▮

4. ESTIMATES ON $\mathcal{F}K$, $\mathcal{T}K$

In this section we obtain estimates on the size of the polymer activities K under each of the renormalization group steps $K \to \mathcal{F}K$, $K \to \mathcal{T}K$. The estimate on $K \to \mathcal{F}K$ is based on similar results of Brydges and Yau[1] but now generalized to take into account the dependence on the external field ρ. The estimate on $\mathcal{T}K$ is a generalization of a result in ref. 2.

In the following we often want to take the functional derivative of a product of functions. For the product functional $K(\rho) = K^1(\rho) \cdots K^N(\rho)$, where K^i is defined on $\rho \in C'(\Lambda)$ and has derivatives given by bounded Borel functions, we have that K is differentiable and the pth derivative is given by the formula

$$K_p(\mathbf{x}) = \sum_{\pi \in P(N,p)} \prod_{i=1}^{N} K^i_{|\pi(i)|}(\mathbf{x}_{\pi(i)}) \tag{74}$$

Here the sum is over ordered partitions of $\{1,..., p\}$ into at most N sets, i.e., $\pi \in P(N, p)$ is a map from $\{1,..., N\}$ to subsets of $\{1,..., p\}$ (possibly empty) such that $\pi(i) \cap \pi(j) = \varnothing$ for $i \neq j$ and $\bigcup_i \pi(i) = \{1,..., p\}$. For $\mathbf{x} = (x_1,..., x_p)$ we have defined $\mathbf{x}_{\pi(i)} = (x_r)_{r \in \pi(i)}$.

For a functional $K(\psi) = \prod_{i=1}^{N} K^i(\psi)$ with K^i defined on $\psi = (\psi_1, \psi_2) \in C(\Lambda \times \Omega_1) \times C(\Lambda \times \Omega_2)$ with derivatives given by measures, we have that K is differentiable and the derivative of order n_1, n_2 is given by the measure

$$K_{n_1,n_2}(\xi^1, \xi^2) = \sum_{\substack{\nu_1 \in P(N,n_1) \\ \nu_2 \in P(N,n_2)}} \prod_{i=1}^{N} K^i_{|\nu_1(i)|,\,|\nu_2(i)|}(\xi^1_{\nu_1(i)}, \xi^2_{\nu_2(i)}) \tag{75}$$

where $\xi^\alpha = (\xi_1^\alpha, ..., \xi_{n_\alpha}^\alpha) \in (\Lambda \times \Omega_\alpha)^{n_\alpha}$, $\alpha = 1, 2$. (Here we have written the measures as if they were functions.) We abbreviate (75) as

$$K_n(\xi) = \sum_{\nu \in P(N,n)} \prod_{i=1}^{N} K^i_{|\nu(i)|}(\xi_{\nu(i)}) \tag{76}$$

For a function $K(\psi, \rho)$ of both variables a similar product formula holds.

Proposition 1. Let C be the covariance of a Gaussian measure and suppose a continuous family of large field regulators $g(t)$, $0 \leqslant t \leqslant 1$, satisfies the homotopy property

$$\mu_{(t-s)C} * g(s) \leqslant g(t), \qquad 0 \leqslant s < t \leqslant 1 \tag{77}$$

and Eq. (83) below. Suppose $\Gamma(X, \mathbf{x})$ satisfies (53) and (54). Let $h = (h, h)$, $h' = (h', h')$, $h' < h$, be given and let $K = K(X, \psi, \rho)$ be a functional such that

$$\|K\|_{g(0), \Gamma, h, u} \leqslant \frac{(h - h')^2}{16 \, \|C_\theta\|} \tag{78}$$

where $\|C_\theta\|$ is given by (87). Then there is a functional $\mathscr{F}K = \mathscr{F}K(X, \psi, \rho)$ such that $\mathscr{E}xp(\Box + \mathscr{F}K) = \mu_c * \mathscr{E}xp(\Box + K)$, and

$$\|\mathscr{F}K\|_{g(1), \Gamma, h', u} \leqslant \|K\|_{g(0), \Gamma, h, u} \tag{79}$$

Proof. We have $\mathscr{F}K = K(1)$, where $K(t)$ is given by (26). In detail this says

$$K(t, X) = \mu_{tC} * K(X) + \tfrac{1}{2} \int_0^t ds \sum_{X_1, X_2} \sum_{a_1, a_2} \int d\xi_1 \, d\xi_2$$
$$\times C(\xi_1, \xi_2) \, \mu_{(t-s)C} * [K_{a_1}(s, X_1, \xi_1) \, K_{a_2}(s, X_2, \xi_2)] \tag{80}$$

Here the sums are over disjoint pairs X_1, X_2 with $X_1 \cup X_2 = X$, over $a_i = (1, 0)$ or $(0, 1)$, and over ξ_i in $(\Lambda \times \Omega^1)$ or $(\Lambda \times \Omega^2)$, depending on a_i.

Taking derivatives, we have

$$K_{n,p}(t, X, \xi, \mathbf{x})$$
$$= \mu_{tC} * K_{n,p}(X, \xi, \mathbf{x})$$
$$+ \tfrac{1}{2} \int_0^t ds \sum_{X_1, X_2} \sum_{a_1, a_2} \sum_{\nu \in P(2,n)} \sum_{\pi \in P(2,p)} \int d\xi_1' \, d\xi_2' \, C(\xi_1', \xi_2')$$
$$\times \mu_{(t-s)C} * [K_{a_1 + |\nu(1)|, |\pi(1)|}(s, X_1, \xi_1', \xi_{\nu(1)}, \mathbf{x}_{\pi(1)})$$
$$\times K_{a_2 + |\nu(2)|, |\pi(2)|}(s, X_2, \xi_2', \xi_{\nu(2)}, \mathbf{x}_{\pi(2)})] \tag{81}$$

[Note: $(\xi_i', \xi_{\nu(i)})$ is understood to be arranged so variables in $\Lambda \times \Omega^1$ precede variables in $\Lambda \times \Omega^2$.] Inserting characteristic functions, and using the consequence of the property (77),

$$\|\mu_{(t-s)C} * A\|_{g(t)} \leqslant \|A\|_{g(s)} \tag{82}$$

and since $X_1 \cap X_2 = \varnothing$,

$$g(s, X) = g(s, X_1)\, g(s, X_2) \tag{83}$$

we find

$$\begin{aligned} \|K_{n,p}(t, X; \mathbf{x})\, 1_\Delta\|_{g(t)} \leqslant{}& \|K_{n,p}(X; \mathbf{x})\, 1_\Delta\|_{g(0)} \\ &+ \tfrac{1}{2}\int_0^t ds \sum_{X_1, X_2} \sum_{a_1, a_2} J \sum_{\nu, \pi} \sum_{\Delta_1', \Delta_2'} C(\Delta_1', \Delta_2') \\ &\times \prod_{i=1,2} \|K_{a_i + |\nu(i)|, |\pi(i)|}(s, X_i, \mathbf{x}_{\pi(i)})\, 1_{\Delta_i' \times \Delta_{\nu(i)}}\|_{g(s)} \end{aligned} \tag{84}$$

where

$$C(\Delta_1, \Delta_2) = \sup_{|\alpha_i| \leqslant 2} \sup_{x_i \in \Delta_i} |(\partial^{\alpha_1} \partial^{\alpha_2} C)(x_1, x_2)| \tag{85}$$

Next we do the sum over X and Δ. Since $X_i \ni \Delta_i'$, we may use from (54)

$$\Gamma(X, \mathbf{x}) \leqslant \Gamma(X_1, \mathbf{x}_{\pi(1)})\, \Gamma(X_2, \mathbf{x}_{\pi(2)})\, \theta(d(\Delta_1', \Delta_2')) \tag{86}$$

We define

$$\|C_\theta\| = \sup_{\Delta_1} \sum_{\Delta_2} C(\Delta_1, \Delta_2)\, \theta(d(\Delta_1, \Delta_2)) \tag{87}$$

Then we claim that

$$\begin{aligned} \|K_{n,p}(t)\|_{g(t),\Gamma} \leqslant{}& \|K_{n,p}\|_{g(0),\Gamma} \\ &+ \int_0^t ds \sum_{a_1, a_2} \sum_{\nu, \pi} \|C_\theta\| \prod_{i=1,2} \|K_{a_i + |\nu(i)|, |\pi(i)|}(s)\|_{g(s),\Gamma} \end{aligned} \tag{88}$$

For $p = 0$ the argument is given in ref. 1. For $p > 0$ we generalize as follows, taking into account the dichotomy in (49). Either $|\pi(1)| \neq 0$ or $|\pi(2)| \neq 0$; suppose $|\pi(1)| \neq 0$, but possibly $|\pi(2)| = 0$. Then we sum over $\{X_2 : X_2 \ni \Delta_2'\}$ and $\Delta_{\nu(2)}$ (but not Δ_2') to identify something dominated by $\|K_{a_2 + |\nu(2)|, |\pi(2)|}\|_{g(s),\Gamma}$. Next do the sum over Δ_2' to identify $\|C_\theta\|$. Finally, sum over X_1, $\Delta_{\nu(1)}$, Δ'_1 to get $\|K_{a_1+|\nu(1)|,|\pi(1)|}\|_{g(s),\Gamma}$ [no pin is required, since $|\pi(1)| \neq 0$].

Now the sums over partitions into two sets only depend on the number of elements in each set and we simplify to

$$\|K_{n,p}(t)\|_{g(t),\Gamma} \leqslant \|K_{n,p}\|_{g(0),\Gamma} + \|C_\theta\| \int_0^t ds \sum_{a_1,a_2} \sum_{n_1+n_2=n} \sum_{p_1+p_2=p} \frac{n!}{n_1!\,n_2!} \frac{p!}{p_1!\,p_2!} \times \prod_{i=1,2} \|K_{a_i+n_i,p_i}(s)\|_{g(s),\Gamma} \tag{89}$$

Multiplying by $h^n/n!$ and $u^p/p!$ and summing over n, p yields

$$\|K(t)\|_{g(t),\Gamma,h,u} \leqslant \|K\|_{g(0),\Gamma,h,u} + \|C_\theta\| \int_0^t ds\, [\partial/\partial h\, \|K(s)\|_{g(s),\Gamma,h,u}]^2 \tag{90}$$

This differential inequality gives the result by ref. 1, Lemma 8.4.

Proposition 2. Let $K(X, \psi, \rho)$ be a local analytic functional and let

$$\mathcal{T}K(X, \psi, \rho) = K(X, \psi + i(\partial C\rho, \partial^2 C\rho), \rho) \tag{91}$$

Then $\mathcal{T}K$ is a local analytic functional and

$$\|\mathcal{T}K\|_{G,\Gamma,h,u} \leqslant \|K\|_{G,\Gamma,h+\gamma\|C_\theta\|u,u} \tag{92}$$

where $\gamma = \sup_{x \in \Lambda} \Gamma(\varnothing, x)$ and $\|C_\theta\|$ is given by (87).

Proof. We compute, for $p \neq 0$,

$$(\mathcal{T}K)_{n,p}(X; \xi, \mathbf{x}) = \sum_{\pi \in P(3,p)} \int K_{n+|\pi'|,|\pi(0)|}(X, (\xi, \xi'), \mathbf{x}_{\pi(0)}) \prod_{r \in \pi'} iC(\xi'_r, x_r)\, d\xi' \tag{93}$$

Here $\pi = (\pi(0), \pi')$, $\pi' = (\pi(1), \pi(2))$. We have $\xi' = (\xi'_1, \xi'_2)$ with $\xi'_i \in (\Lambda \times \Omega_i)^{\pi(i)}$. Also, (ξ, ξ') really means $((\xi_1, \xi'_1), (\xi_2, \xi'_2))$.

Inserting characteristic functions and taking the norm yields

$$\|(\mathcal{T}K)_{n,p}(X; \mathbf{x})\, 1_\Delta\|_G \leqslant \sum_\pi \sum_{\Delta'} \|K_{n+|\pi'|,|\pi(0)|}(X; \mathbf{x}_{\pi(0)})\, 1_{\Delta \times \Delta'}\|_G \prod_{r \in \pi'} C(\Delta'_r, \Delta_{\mathbf{x}_r}) \tag{94}$$

where $\Delta' = \{\Delta'_r : r \in \pi'\}$. By (54), the large set regulator satisfies

$$\Gamma(X, \mathbf{x}) \leqslant \Gamma(X, \mathbf{x}_{\pi(0)}) \left[\prod_{r \in \pi'} \gamma\theta(d(\Delta'_r, \Delta_{\mathbf{x}_r})) \right] \tag{95}$$

since $X \ni \Delta'_r$. Therefore

$$\begin{aligned} \|(\mathscr{T}K)_{n,p}\|_{G,\Gamma} &\leqslant \sum_{\pi} \sup_{\mathbf{x}} \sum_{X, \Delta, \Delta'} \left[\prod_{r \in \pi'} \gamma C_\theta(\Delta'_r, \Delta_{\mathbf{x}_r}) \right] \\ &\quad \times \Gamma(X, \mathbf{x}_{\pi(0)}) \, \|K_{n+|\pi'|, |\pi(0)|}(X, \mathbf{x}_{\pi(0)}) \, 1_{\Delta \times \Delta'}\|_G \\ &\leqslant \sum_{\pi} \gamma^{|\pi'|} \, \|C_\theta\|^{|\pi'|} \, \|K_{n+|\pi'|, |\pi(0)|}\|_{G,\Gamma} \end{aligned} \tag{96}$$

The last step is straightforward for the terms with $\pi(0) \neq \varnothing$ if we use $C_\theta(\Delta', \Delta) \leqslant \|C_\theta\|$. If $\pi(0) = \varnothing$, $\pi'(0) \neq \varnothing$, let s be an element of π'. Keep Δ'_s fixed and sum over X, Δ, $\{\Delta'_r\}_{r \neq s}$. Then $X \ni \Delta'_s$ and so we can identify something dominated by $\|K_{n+|\pi'|, \pi(0)}\|_{G,\Gamma}$. Finally, sum over Δ'_s using $\sum_{\Delta'_s} C_\theta(\Delta'_s, \Delta_{\mathbf{x}_s}) \leqslant \|C_\theta\|$.

We continue with

$$\|(\mathscr{T}K)_{n,p}\|_{G,\Gamma} \leqslant \sum_{\substack{p_0+p_1+p_2 \\ =p}} \frac{p!}{p_0!\, p_1!\, p_2!} (\gamma \, \|C_\theta\|)^{p_1+p_2} \, \|K_{n+p', p_0}\|_{G,\Gamma} \tag{97}$$

Now multiply by $h^n/n!$, $u^p/p!$, sum over n, p, and rearrange the sum to get the answer. ▌

5. ESTIMATES ON $\mathscr{E}K$, $\mathscr{B}K$

In this section we obtain estimates on $\mathscr{E}K$, $\mathscr{B}K$ generalizing similar results of Brydges and Yau,[1] who combine $\mathscr{B}$ and $\mathscr{E}$ into one step $\mathscr{B}\mathscr{E}$. We begin with a precise formula for the extraction operation $K \to \mathscr{E}K$, which yields the modified functional $\mathscr{E}K$ after extracting $\sum_X F(X)$, $F(X) \equiv F_0(X) + \mathscr{E}(X)$ from K.

Let $R(X) = (e^F - 1)(X)$ ($=0$ if $X \notin \mathscr{S}$), let $I(X)$ be any function satisfying $I(X) \doteq K(X) - F(X)$ (i.e., they are equal when $\psi = \psi_\phi$), and define $J(X)$ by

$$J(X) = I(X) + F(X) - \sum_{\{X_i\} \to X} \prod_i R(X_i) \tag{98}$$

The sum above is defined to be empty if X is not connected, and to be over all collections $\{X_i\}$ of distinct polymers with $\bigcup_i X_i = X$ if X is connected. The condition on $\mathscr{E}K$ taken from ref. 1, Eq. (2.6) is:

$$\mathscr{E}xp(\square + \mathscr{E}K) = \sum_X (\mathscr{E}xp\, J)(X) \exp\left[- \sum_{Y:\, Y \cap X \neq \varnothing} F(Y) \right] \tag{99}$$

We expand $\mathscr{E}xp\, J$ and perform a Mayer expansion on $\exp[-\sum_Y F(Y)]$ to obtain the condition

$$\mathscr{E}xp(\square + \mathscr{E}K) = \sum_{\{X_i\}} \sum_{\{Y_j\}} \prod_i J(X_i) \prod_j (e^{-F} - 1)(Y_j) \tag{100}$$

where the $\{X_i\}$ are disjoint, the $\{Y_j\}$ are distinct elements of $\mathscr{S}$, and each Y_j intersects some X_i. Grouping together sets that intersect, we fulfill this condition if $\mathscr{E}K$ is defined by

$$\mathscr{E}K(X) = \sum_{\{X_i\}, \{Y_j\} \to X} \prod_i J(X_i) \prod_j (e^{-F} - 1)(Y_j) \tag{101}$$

The sum here is over collections $\{X_i\}$, $\{Y_j\}$ which satisfy the above conditions, $(\bigcup X_i) \cup (\bigcup Y_j) = X$, and whose overlap graph is connected.

To complete the specification of $\mathscr{E}K$, we must specify $I = I(X, \psi, \rho)$, so that $I \doteq K - F$ and yet is irrelevant. If $X \notin \mathscr{S}$, then $F, R = 0$ and we take $I = K$. We write

$$K(X, \psi, \rho) = K_0(X, \psi) + K_{p>0}(X, \psi, \rho) \tag{102}$$

where $K_0 = K(\rho = 0)$, and similarly $I = I_0 + I_{p>0}$. Then, for $X \in \mathscr{S}$ we define

$$I_0(X, \psi) = (K_0(X, \psi) - K_{n \leqslant 2,0}(X, \psi)) + W(X, \psi) \tag{103}$$

where $K_{n \leqslant 2,0}(X, \psi)$ is the expansion of K_0 up to second order in ψ,

$$K_{n \leqslant 2,0}(X, \psi) = K_{0,0}(X, 0) + \tfrac{1}{2} \sum_{\mu, \nu} \int dx\, dy\, K_{2,0}(X, 0; x, \mu, y, \nu)\, \psi_\mu(x)\, \psi_\nu(y) \tag{104}$$

We need $W \doteq K_{n \leqslant 2,0} - F_0$, and we choose

$$W(X, \psi) = (2\, |X|)^{-1} \sum_{\mu, \nu, \sigma} \int_0^1 ds \int_X dz \int dx\, dy\, K_{2,0}(X, 0; x, \mu, y, \nu) \times [\dot{\gamma}_{xz,\sigma}(s)\, \psi_{\mu\sigma}(\gamma_{xz}(s))\, \psi_\nu(\gamma_{yz}(s)) + \dot{\gamma}_{yz,\sigma}(s)\, \psi_\mu(\gamma_{xz}(s))\, \psi_{\nu\sigma}(\gamma_{yz}(s))] \tag{105}$$

where γ_{xz} is a curve from z to x lying entirely in X defined in some standard fashion. Lastly, we define

$$I_{p>0}(X, \psi, \rho) = K_{p>0}(X, \psi, \rho) - \mathscr{E}(X, \rho) \tag{106}$$

To estimate $\mathscr{E}K$, $\mathscr{B}K$ we will use the following result. Let $\mathscr{M}(N, X, \tau)$ be the set of all ordered N-tuples $(X_1,..., X_N)$ of nonempty subsets of X such that:

1. $\bigcup_i X_i = X$.
2. At most τ of the X_i overlap.
3. The overlap graph for $X_1,..., X_N$ [i.e., all pairs (i, j) so $X_i \cap X_j \neq \varnothing$] is connected.

Lemma 3. Let $\mathscr{M} \subset \mathscr{M}(N, X, \tau)$ and

$$K(X) = \sum_{(X_1,..., X_N) \in \mathscr{M}} \prod_{i=1}^{N} K^i(X_i) \tag{107}$$

Then for $\gamma > 1$ and $\varepsilon > 0$

$$\|K\|_{G(\varepsilon), \Gamma, h, u} \leqslant N!\, (4 \cdot 3^d/\log \gamma)^{N-1} \prod_{i=1}^{N} \|K^i\|_{G(\varepsilon/\tau), \gamma\Gamma, h, u} \tag{108}$$

where $(\gamma\Gamma)(X, \mathbf{x}) = \gamma^{|X|}\Gamma(X, \mathbf{x})$.

Proof. (Compare ref. 1, Lemma 5.1.) Taking derivatives, we have

$$K_{n,p}(X, \xi, \mathbf{x}) = \sum_{\substack{\nu \in P(n,N) \\ \pi \in P(p,N)}} \sum_{(X_1,..., X_N) \in \mathscr{M}} \prod_i K^i_{|\nu(i)|, |\pi(i)|}(X_i, \xi_{\nu(i)}, \mathbf{x}_{\pi(i)}) \tag{109}$$

Lemma 9 in the Appendix shows that

$$G(\varepsilon, X)^{-1} \leqslant \prod_i G(\varepsilon/\tau, X_i)^{-1} \tag{110}$$

We take the measure norm of (109) on Δ, multiply by this inequality, and take the supremum over fields to obtain

$$\|K_{n,p}(X; \mathbf{x})\, 1_\Delta\|_{G(\varepsilon)} \leqslant \sum_{\nu, \pi} \sum_{(X_1,..., X_N)} \prod_i \|K^i_{|\nu(i)|, |\pi(i)|}(X_i, \mathbf{x}_{\pi(i)})\, 1_{\Delta_{\nu(i)}}\|_{G(\varepsilon/\tau)} \tag{111}$$

Next, by the overlap connectedness, we have

$$\Gamma(X, \mathbf{x}) \leqslant \prod_i \Gamma(X_i, \mathbf{x}_{\pi(i)}) \tag{112}$$

We multiply by this, sum over X and Δ, and sup over $\mathbf{x}$. Then the sum over $(X_1,..., X_N)$ is estimated in a standard way by summing over trees T on $(1,..., N)$ and then summing over $(X_1,..., X_N)$ with overlap graph containing T. We prune the lines of T from the twigs inward, using for each line (i, j) the bound

$$\sum_{\Delta_{\nu(i)}} \sum_{X_i:\, X_i \cap X_j \neq \varnothing} |X_i|^{\delta_i - 1} \Gamma(X_i, \mathbf{x}) \, \|K^i_{|\nu(i)|, \pi(i)}(X_i; \mathbf{x}) \, 1_{\Delta_{\nu(i)}}\|_{G(\varepsilon/\tau)}$$
$$\leqslant (\delta_i - 1)! \, (\log \gamma)^{-(\delta_i - 1)} \, (3^d \, |X_j|) \, \|K^i_{|\nu(i)|, \pi(i)}\|_{G(\varepsilon/\tau), \gamma\Gamma} \tag{113}$$

Here δ_i is the coordination number for i in the graph T and we used

$$|X|^{\delta_i - 1} \leqslant (\delta_i - 1)! \, (\log \gamma)^{-(\delta_i - 1)} \, \gamma^{|X|} \tag{114}$$

For $p > 0$, we leave until last the sum over X_{i_0}, where i_0 is chosen so that $\pi(i_0) \neq \varnothing$. In the last step no pin is needed and we just get

$$\delta_{i_0}! \, (\log \gamma)^{-\delta_{i_0}} \, \|K^{i_0}_{|\nu(i_0)|, \pi(i_0)}\|_{G(\varepsilon/\tau), \gamma\Gamma}$$

Thus we have

$$\|K_{n,p}\|_{G(\varepsilon), \Gamma} \leqslant \sum_{\nu, \pi, T} \delta_{i_0}! \, (\log \gamma)^{-\delta_{i_0}} \prod_{i \neq i_0} [(\delta_i - 1)! \, (\log \gamma)^{-(\delta_i - 1)} \, 3^d]$$
$$\times \prod_i \|K^i_{|\nu(i)|, |\pi(i)|}\|_{G(\varepsilon/\tau), \gamma\Gamma} \tag{115}$$

Now $\delta_{i_0}! \leqslant (N-1)(\delta_{i_0} - 1)!$ and $\sum_T \prod_i (\delta_i - 1)! \leqslant (N-2)! \, 4^{N-1}$ (by Cayley's theorem) and $\sum_i (\delta_i - 1) = N - 2$. Thus,

$$\|K_{n,p}\|_{G(\varepsilon), \Gamma} \leqslant (N-1)! \, (3^d \cdot 4/\log \gamma)^{N-1} \sum_{\nu, \pi} \prod_i \|K^i_{|\nu(i)|, |\pi(i)|}\|_{G(\varepsilon/\tau), \gamma\Gamma}$$
$$\leqslant N! \, (4 \cdot 3^d/\log \gamma)^{N-1} \sum_{\sum n_i = n, \sum p_i = p} n! \, p!$$
$$\times \prod_i \frac{1}{n_i! \, p_i!} \|K^i_{n_i, p_i}\|_{G(\varepsilon/\tau), \gamma\Gamma} \tag{116}$$

For $p = 0$ the argument is similar.[1] Now multiply by $(h^n/n!)(u^p/p!)$ and sum to complete the proof. ∎

Lemma 4. (a) There are constants c, C such that for any g

$$\|F\|_{G(\varepsilon), \Gamma, h, u} \leqslant C(1 + 1/\varepsilon h_1^2) \, \|K\|_{g, \Gamma, h, u}$$

(b) If $\|K\|_{g,\Gamma,h,u} \leqslant c(\varepsilon h_1^2/1+\varepsilon h_1^2)$, then

$$\begin{aligned} \|e^{\pm F}-1\|_{G(\varepsilon),\Gamma,h,u} &\leqslant C(1+1/\varepsilon h_1^2)\,\|K\|_{g,\Gamma,h,u} \leqslant cC \\ \|e^{\pm F}-1\mp F\|_{G(\varepsilon),\Gamma,h,u} &\leqslant C(1+1/\varepsilon h_1^2)^2\,\|K\|^2_{g,\Gamma,h,u} \leqslant c^2 C \end{aligned} \tag{117}$$

Proof. (a) The bound on $F_0(X,\psi)=F(X,\psi,0)$ is proved in ref. 1, Lemma 4.1. For $F_{p>0}=\mathscr{E}$ we simply note

$$\mathscr{E}_p(X,\mathbf{x}) = \begin{cases} K_{0,p}(X,0,\mathbf{x}) & \text{if } \; p \neq 0 \text{ and } X \cup \delta_{\mathbf{x}} \in \mathscr{S} \\ 0 & \text{otherwise} \end{cases} \tag{118}$$

and hence $\|\mathscr{E}\| \leqslant \|K\|$.

(b) This follows from (a) as in ref. 1, Lemma 4.2. ▮

Proposition 3. Let $\varepsilon h_1^2 \geqslant 1$, $0<\delta\varepsilon<\varepsilon$, and $1<\gamma\leqslant 2$. Then there are constants c, C_1, C_2, C_3, so that if

$$\|K\|_{G(\varepsilon),\gamma^2\Gamma,h,u} \leqslant c \log\gamma\,[\delta\varepsilon\, h_1^2/(\tau+\delta\varepsilon\, h_1^2)] \tag{119}$$

then:

(a) $\|I\|_{G(\varepsilon),\Gamma,h,u} \leqslant C_1\,\|K\|_{G(\varepsilon),\Gamma,h,u}$

(b) $\|J-I\|_{G(\varepsilon),\Gamma,h,u} \leqslant (C_2/\log\gamma)(1+\tau/\varepsilon h_1^2)^2\,\|K\|^2_{G(\varepsilon),\gamma\Gamma,h,u}$

(c) $\|\mathscr{E}K-J\|_{G(\varepsilon+\delta\varepsilon),\Gamma,h,u} \leqslant (C_3/\log\gamma)(1+\tau/\delta\varepsilon h_1^2)^2\,\|K\|^2_{G(\varepsilon),\gamma^2\Gamma,h,u}$

Here τ is the maximum number of distinct small sets with nonempty intersection.

Remarks. Combining (a)–(c) gives a bound on $\|\mathscr{E}K\|$. The quantities $\mathscr{E}K-J$ and $J-I$ are seen to be of $\mathscr{O}(\|K\|^2)$ and hence small. However, the crude estimate (a) on the first-order part I does not exploit the cancellations introduced by the extraction operation, and so does not show the required contraction. An improved bound on I using these cancellations will be proved in Section 6.

Proof. (a) The result $\|I_0\| \leqslant \mathscr{O}(1)\,\|K_0\|$ for $I_0(X,\psi)=I(X,\psi,0)$ is just as in ref. 1, Chapter 4. For the ρ derivatives of I $(p>0)$ we have

$$I_{n,p}(X,\phi;\mathbf{x}) = \begin{cases} K_{n\,p}(X,\phi;\mathbf{x}) & \text{if } \; n>0 \text{ or } X\cup\delta_{\mathbf{x}} \notin \mathscr{S} \\ K_{0,p}(X,\phi;\mathbf{x}) - K_{0,p}(X,0;\mathbf{x}) & \text{otherwise} \end{cases} \tag{120}$$

Thus $\|I_{p>0}\| \leqslant 2\,\|K_{p>0}\|$.

(b) Recall $J-I=(F-R)-R^+$, where $R=e^F-1$ and

$$R^+(X) = \sum_{N=2}^{\infty} 1/N! \sum_{(X_1,\ldots,X_N)} \prod_i R(X_i) \tag{121}$$

Here the sum is over $(X_1,..., X_N) \in \mathscr{M}(X, N, \tau)$ with $X_i \neq X_j$ and $X_i \in \mathscr{S}$. By Lemma 4b we have

$$\|R\|_{G(\varepsilon/\tau), \gamma\Gamma, h, u} \leqslant C(1 + \tau/\varepsilon h_1^2) \|K\|_{G(\varepsilon), \gamma\Gamma, h, u} \tag{122}$$

$$\|F - R\|_{G(\varepsilon), \Gamma, h, u} \leqslant C(1 + \tau/\varepsilon h_1^2)^2 \|K\|^2_{G(\varepsilon), \Gamma, h, u} \tag{123}$$

By Lemma 3

$$\|R^+\|_{G(\varepsilon), \Gamma, h, u} \leqslant \sum_{N=2}^{\infty} (4 \cdot 3^d/\log \gamma)^{N-1} \|R\|^N_{G(\varepsilon/\tau), \gamma\Gamma, h, u} \tag{124}$$

Now using (122) we find

$$\|R^+\|_{G(\varepsilon/\tau), \gamma\Gamma, h, u} \leqslant (2 \cdot 4 \cdot 3^d/\log \gamma)\, C^2(1 + \tau/\varepsilon h_1^2)^2 \|K\|^2_{G(\varepsilon), \gamma\Gamma, h, u} \tag{125}$$

provided c is taken small enough so that the infinite series is dominated by twice the $N=2$ term. The required bound on $\|J - I\|$ (with a new constant C_2) follows when the bound on R^+ is combined with (123).

(c) We write

$$(\mathscr{E}K - J)(X) = \sum_{\substack{N \geqslant 1, M \geqslant 0 \\ N + M \geqslant 2}} \frac{1}{N!\, M!} \sum_{(X_1,..., X_N)} \sum_{(Y_1,..., Y_M)} \prod_{i=1}^{N} J(X_i) \times \prod_{j=1}^{M} (e^{-F} - 1)(Y_j) \tag{126}$$

where the sum is over sets $(X_1,..., X_N, Y_1,..., Y_M)$ in $\mathscr{M}(X, N+M, \tau)$ such that the X_i are disjoint, the Y_j are distinct elements in $\mathscr{S}$, and each Y_j intersects some X_i. Thus, we may apply Lemma 3 again. However, first note that because of the additional restrictions on X_i, Y_j we may replace (110) by

$$\begin{aligned} G(\varepsilon + \delta\varepsilon, X)^{-1} &= G(\varepsilon, X)^{-1}\, G(\delta\varepsilon, X)^{-1} \\ &\leqslant \prod_{i=1}^{N} G(\varepsilon, X_i)^{-1} \prod_{j=1}^{M} G(\delta\varepsilon/\tau, Y_j)^{-1} \end{aligned} \tag{127}$$

With this modification we have

$$\begin{aligned} &\|\mathscr{E}K - J\|_{G(\varepsilon + \delta\varepsilon), \Gamma, h, u} \\ &\quad \leqslant \sum_{\substack{N \geqslant 1, M \geqslant 0 \\ N + M \geqslant 2}} \frac{(N+M)!}{N!\, M!} \left(\frac{4 \cdot 3^d}{\log \gamma}\right)^{N+M-1} \\ &\quad\quad \times \|J\|^N_{G(\varepsilon), \gamma\Gamma, h, u} \|e^{-F} - 1\|^M_{G(\delta\varepsilon/\tau), \gamma\Gamma, h, u} \end{aligned} \tag{128}$$

Combining (a) and (b) yields

$$\|J\|_{G(\varepsilon),\gamma\Gamma,h,u} \leqslant 2C_1 \|K\|_{G(\varepsilon),\gamma^2\Gamma,h,u} \tag{129}$$

and by Lemma 4b

$$\|e^{-F}-1\|_{G(\delta\varepsilon/\tau),\gamma\Gamma,h,u} \leqslant C(1+\tau/\delta\varepsilon\, h_1^2)\, \|K\|_{G(\varepsilon),\gamma\Gamma,h,u} \tag{130}$$

Insertion of these two inequalities into (128) leads to the required bound provided c is small enough that the infinite series is bounded by twice the $N+M=2$ terms. ∎

Now we turn to the estimate on $\mathscr{B}K$ defined in (34). The natural estimate for $\mathscr{B}K$ is in terms of a new norm for K [cf. ref. 1, Eq. (3.2)]:

$$\|K\|^{(1)}_{G,\Gamma,h,u} = \sum_{n,p} h^n u^p/n!\; p!\; \|K_{n,p}\|^{(1)}_{G,\Gamma} \tag{131}$$

where

$$\|K_{n,p}\|^{(1)}_{G,\Gamma} = \begin{cases} \sup\limits_{\Delta_0^{(1)}} \sum\limits_{\Delta,X:\, \bar{X}\ni\Delta_0^{(1)}} \Gamma(\bar{X})\, \|K_{n,0}(X)\, 1_\Delta\|_G & p=0 \\ \sup\limits_{\mathbf{x}} \sum\limits_{\Delta,X} \Gamma(\bar{X},\mathbf{x})\, \|K_{n,p}(X,\mathbf{x})\, 1_\Delta\|_G & p\neq 0 \end{cases} \tag{132}$$

Here Γ is a large set regulator on L-sets and $\Delta_0^{(1)}$ is an L-block.

Proposition 4. If $0<\delta<1$ and

$$\|K\|^{(1)}_{G,\gamma\Gamma,h,u} \leqslant \delta(\log\gamma/4\cdot 3^d) \tag{133}$$

then

$$\|\mathscr{B}K\|_{G,\Gamma,h,u} \leqslant (1-\delta)^{-1}\, \|K\|^{(1)}_{G,\gamma\Gamma,h,u} \tag{134}$$

Proof. The definition of $\mathscr{B}K$ almost fits in the framework of Lemma 3 and we follow that proof, noting the differences. Since the X_i do not intersect, we may replace (110) by $G(U)^{-1}\leqslant\prod_i G(X_i)^{-1}$, by taking Lemma 9 in the Appendix with $\tau=1$. The connectedness condition is on the $\bar{X}_i$ so we use $\Gamma(U,\mathbf{x})\leqslant\prod_i \Gamma(\bar{X}_i,\mathbf{x}_{\pi(i)})$ instead of (112). Instead of (113), we have

$$\begin{aligned} &\sum_{\Delta_{\nu(i)}} \sum_{X_i:\, \bar{X}_i\cap\bar{X}_j\neq\varnothing} |\bar{X}_i|^{\delta_i-1}\, \Gamma(\bar{X}_i,\mathbf{x})\, \|K_{|\nu(i)|,|\pi(i)|}(X_i;\mathbf{x})\, 1_{\Delta_{\nu_i}}\|_G \\ &\qquad \leqslant (\delta_i-1)!\, (\log\gamma)^{-(\delta_i-1)}\, (3^d\, |\bar{X}_j|)\, \|K_{|\nu(i)|,|\pi(i)|}\|^{(1)}_{G,\gamma\Gamma} \end{aligned} \tag{135}$$

This yields

$$\|\mathscr{B}K\|_{G,\Gamma,h,u} \leqslant \sum_{N=1}^{\infty} (4\cdot 3^d/\log\gamma)^{N-1} (\|K\|^{(1)}_{G,\gamma\Gamma,h,u})^N$$
$$\leqslant (1-\delta)^{-1} \|K\|^{(1)}_{G,\gamma\Gamma,h,u} \quad \blacksquare$$

Proposition 5. Let $\Gamma^{(1)}$ and Γ be large set regulators which satisfy the inequality (66) for some $\gamma \geqslant 1$. Then, for any K,

$$\|K_{p=0} 1_{\mathscr{S}}\|^{(1)}_{G,\Gamma^{(1)},h} \leqslant c(\gamma)\, L^d\, \|K_{p=0} 1_{\mathscr{S}}\|_{G,\gamma^{-1}\Gamma,h} \tag{136a}$$

$$\|K_{p>0} 1_{\mathscr{S}}\|^{(1)}_{G,\Gamma^{(1)},h,u} \leqslant c(\gamma)\, L^{d/2-1}\, \|K_{p>0} 1_{\mathscr{S}}\|_{G,\gamma^{-1}\Gamma,h,u} \tag{136b}$$

$$\|K 1_{\not\mathscr{S}}\|^{(1)}_{G,\Gamma^{(1)},h,u} \leqslant c(\gamma)\, L^{-1}\, \|K 1_{\not\mathscr{S}}\|_{G,\gamma^{-1}\Gamma,h,u} \tag{136c}$$

Remark. We use the notation $1_{\mathscr{S}}$ to mean $1_{X\cup\delta_x\in\mathscr{S}}$ and $1_{\not\mathscr{S}}$ for $1_{X\cup\delta_x\notin\mathscr{S}}$.

Proof. For (a)

$$\|K_{p=0} 1_{\mathscr{S}}\|^{(1)}_{G,\Gamma^{(1)},h} = \sup_{\Delta_0^{(1)}} \sum_{\Delta,X\in\mathscr{S};\bar{X}\supset\Delta_0^{(1)}} \Gamma^{(1)}(\bar{X})\, \|K_{p=0}(X)\, 1_\Delta\|_{G,h}$$
$$\leqslant c(\gamma) \sup_{\Delta_0^{(1)}} \sum_{\Delta_0\subset\Delta_0^{(1)}} \sum_{\Delta,X\in\mathscr{S};X\supset\Delta_0} (\gamma^{-1}\Gamma)(X)\, \|K_{p=0}(X)\, 1_\Delta\|_{G,h}$$
$$\leqslant c(\gamma)\, L^d\, \|K_{p=0} 1_{\mathscr{S}}\|_{G,\gamma^{-1}\Gamma,h}$$

In the second line above, the sum over Δ_0 is over unit blocks of $\Delta_0^{(1)}$. The proofs of (b) and (c) are similar. For (b) there is no factor L^d, since there is no pin for the sum over X. $\blacksquare$

6. THE MAIN THEOREM

We are now ready to state and prove bounds on the functionals $K^j(X, \psi, \rho)$. These will be used in Section 7 to determine bounds on the correlation functions.

Throughout this section, η is a parameter with $1/2 \leqslant \eta < 1$ and C_η stands for constants which satisfy $C_\eta \to 1$ as $\eta \to 1$.

We describe the specific norms used to measure the quantities K^j. The large-field regulator has the form $G^j(X, \phi) = G_{\kappa^j}(LX, L^{-(d/2)+1}\varphi(\cdot/L))$ from (47), with

$$\kappa^j = \kappa^0 \left(\sum_{i=0}^{j} 2^{-i}\right) \tag{137}$$

Then $\kappa^0 \leqslant \kappa^j \leqslant 2\kappa^0$ and we choose κ^0 small as specified below.

The large-set regulator Γ^j has the form (56):

$$\Gamma^j(X, \mathbf{x}) = L^{(d/2-1)N^j(\mathbf{x})}\Gamma(X \cup \delta_{\mathbf{x}}) \tag{138}$$

where $\Gamma(X) = A^{|X|}\Theta(X)$ satisfies (65) with $\gamma = \eta^{-1}$. We will also consider $\eta^\alpha \Gamma^j$ defined by

$$(\eta^\alpha \Gamma^j)(X, \mathbf{x}) = \eta^{\alpha |X|}\Gamma^j(X, \mathbf{x}) \tag{139}$$

These also satisfy (53), (54).

The ψ-derivative weights are independent of j and are taken to be

$$h = h_0(1, 1) \tag{140}$$

where $h_0 = aL^{(d/2)+1}$ and a is arbitrary, except that we want $a = \mathcal{O}((1-\eta)^{-1})$ if $\eta \to 1$.

Finally, the ρ-derivative weights are taken to be

$$u^j = L^{-(d/2-1)j}u^0$$
$$u^0 \leqslant (1-\eta)\, h^0(\Gamma(\Delta)\, \|C_\theta\|)^{-1}$$

Theorem 1. Fix η and let L be sufficiently large, or fix $L > 2^d$ and let η be sufficiently close to 1. Then for z real, $|z|$ sufficiently small (depending on L, η), and $\delta^0 = 6\,|z|\, e^{h_0}\Gamma^0(\Delta)$ we have

$$\|K^j\|_{G^j, \Gamma^j, h, u^j} \leqslant \delta^j \equiv (C/L)^j\, \delta^0 \tag{141}$$

for some constant $C = C_\eta$ with $C_\eta/L < 1/2$.

Proof. The proof is by induction on j. For $j = 0$ the functional K^j is independent of ρ and we use the result of Brydges and Yau[1] (see also ref. 2). We have

$$\|K^0\|_{G^0, \Gamma, h, u^0} \leqslant \delta^0 \tag{142}$$

provided $|z|$ is small enough that $\delta^0 < (8e)^{-1}$.

Now we assume (141) holds for j and then prove it for $j+1$ through the sequence $K^\# = \mathscr{T}\mathscr{F}K^j$, $K^* = \mathscr{B}\mathscr{E}K^\#$, and $K^{j+1} = \mathscr{S}K^*$.

For the first step we introduce

$$\begin{aligned} G^\#(X, \phi) &= G_{K^j}(X, \varphi) \\ \Gamma^\#(X, \mathbf{x}) &= (\eta\Gamma^j)(X, \mathbf{x}) \end{aligned} \tag{143}$$

We interpolate between G^j and $G^\#$ by

$$g(t, X) = [G^j(X)]^{1-t}\,[\eta^{-|X|}G^\#(X)]^t \tag{144}$$

Then $g(t, X)$ satisfies the homotopy property (77) provided κ^0 and hence κ^j is small enough, as proved in Proposition 6 in the Appendix.

By Proposition 1 we have

$$\begin{aligned} \|\mathscr{F}K^j\|_{G^\#,\Gamma^\#,\eta h,u^j} &= \|\mathscr{F}K^j\|_{g(1),\Gamma^j,\eta h,u^j} \\ &\leqslant \|K^j\|_{g(0),\Gamma^j,h,u^j} \leqslant \delta^j \end{aligned} \tag{145}$$

provided $\delta^j \leqslant (1-\eta)^2 h_0^2(16\,\|C_\theta\|)^{-1}$, and it suffices that δ^0 satisfy the inequality, which gives another condition on $|z|$.

Furthermore, by Proposition 2 we have, for $K^\# = \mathscr{T}(\mathscr{F}K^j)$,

$$\|K^\#\|_{G^\#,\Gamma^\#,\eta^2 h,u^j} \leqslant \|\mathscr{F}K^j\|_{G^\#,\Gamma^\#,\eta h,u^j} \leqslant \delta^j \tag{146}$$

since $\gamma^\# = \eta L^{(d/2-1)j}\Gamma(\Delta)$ and so

$$\eta^2 h_0 + \gamma^\#\ \|C_\theta\|\ u^j = \eta^2 h_0 + \eta\Gamma(\Delta)\ \|C_\theta\|\ u^0 \leqslant \eta h_0 \tag{147}$$

Now define

$$\begin{aligned} G^*(U,\phi) &= G_{\kappa^{j+1}}(U,\varphi) \\ \Gamma^*(U,\mathbf{x}) &= \Gamma^{j+1,(1)}(U,\mathbf{x}) \\ h^* &= h_0(L^{-d/2}, L^{-d/2-1}) \end{aligned} \tag{148}$$

We will use Proposition 3 and the bound on $K^\#$ to show that

$$\|\mathscr{E}K^\#\|^{(1)}_{G^*,\eta^{-1}\Gamma^*,h^*,u^j} \leqslant (C_\eta/L)\,\delta^j \tag{149}$$

Then by scaling for $K^{j+1} = \mathscr{S}K^*$, $G^{j+1} = \mathscr{S}G^*$, and $\Gamma^{j+1} = \mathscr{S}\Gamma^*$ and using Proposition 4 for $K^* = \mathscr{B}(\mathscr{E}K^\#)$, we complete the proof with

$$\begin{aligned} \|K^{j+1}\|_{G^{j+1},\Gamma^{j+1},h,u^{j+1}} &= \|K^*\|_{G^*,\Gamma^*,h^*,u^j} \\ &\leqslant \eta^{-1}\ \|\mathscr{E}K^\#\|^{(1)}_{G^*,\eta^{-1}\Gamma^*,h^*,u^j} \\ &\leqslant (C_\eta/L)\,\delta^j = \delta^{j+1} \end{aligned} \tag{150}$$

[The condition on $\|\mathscr{E}K^\#\|^{(1)}$ for Proposition 4 is easily satisfied given that (149) holds.]

To establish (149), we estimate $\mathscr{E}K^\# - J$, $J - I$, and I separately. First note that $\eta^{-1}\Gamma^*$ and $\eta^2\Gamma^\#$ satisfy (66) since

$$\begin{aligned}(\eta^{-1}\Gamma^*)(\bar{X}, \mathbf{x}) &= (\eta^{-1})^{|\bar{X}|^{(1)}} \Gamma^{j+1,(1)}(\bar{X}, \mathbf{x}) \\ &\leqslant (\eta^{-1})^{|\bar{X}|^{(1)}} [C_\eta(\eta^4\Gamma^j)(X, \mathbf{x})] \begin{cases} L^{d/2-1} & X \cup \delta_{\mathbf{x}} \in \mathscr{S} \\ L^{-3} & X \cup \delta_{\mathbf{x}} \notin \mathscr{S} \end{cases} \\ &= C_\eta(\eta^2\Gamma^\#)(X, \mathbf{x}) \begin{cases} L^{d/2-1} & X \cup \delta_{\mathbf{x}} \in \mathscr{S} \\ L^{-3} & X \cup \delta_{\mathbf{x}} \notin \mathscr{S} \end{cases} \end{aligned} \tag{151}$$

Thus, we may apply Proposition 5. We follow this with Proposition 3, using also $h^* < h^\# \equiv \eta^2 h$ to obtain

$$\begin{aligned} &\|\mathscr{E}K^\# - J\|^{(1)}_{G^*, \eta^{-1}\Gamma^*, h^*, u^j} \\ &\quad \leqslant C_\eta L^d \|\mathscr{E}K^\# - J\|_{G^*, \eta^2\Gamma^\#, h^\#, u^j} \\ &\quad \leqslant C_\eta L^d (C_3/|\log \eta|)(1 + 2^{(j+1)}\tau/\kappa_0 h_0^2)^2 \|K^\#\|^2_{G^\#, \Gamma^\#, h^\#, u^j} \\ &\quad \leqslant (1-\eta) L^{-1}\delta^j \end{aligned} \tag{152}$$

where the last line follows if δ^0 is sufficiently small.

Similarly, we have

$$\|J - I\|^{(1)}_{G^*, \eta^{-1}\Gamma^*, h^*, u^j} \leqslant (1-\eta) L^{-1}\delta^j \tag{153}$$

To complete the proof of (149) and the theorem, we will establish

$$\|I\|^{(1)}_{G^*, \eta^{-1}\Gamma^*, h^*, u^j} \leqslant C_\eta L^{-1}\delta^j \tag{154}$$

Recall that $I = I_0 + I_{p>0}$ is defined so that

$$\begin{aligned} I(X)\, 1(X \cup \delta_{\mathbf{x}} \notin \mathscr{S}) &= K^\#(X)\, 1(X \cup \delta_{\mathbf{x}} \notin \mathscr{S}) \\ I_{p>0}(X, \mathbf{x})\, 1(X \cup \delta_{\mathbf{x}} \in \mathscr{S}) &= (K^\#_{p>0} - \mathscr{E}_{p>0})(X, \mathbf{x})\, 1(X \cup \delta_{\mathbf{x}} \in \mathscr{S}) \\ I_0(X)\, 1(X \in \mathscr{S}) &= [(K^\#_0 - K^\#_{n \leqslant 2, 0})(X) + W(X)]\, 1(X \in \mathscr{S}) \end{aligned}$$

We estimate separately $I1_{\mathscr{S}}$, $I_{p>0}1_{\mathscr{S}}$, and $I_0 1_{\mathscr{S}}$. By Proposition 5 and since $(G^*)^{-1} < (G^\#)^{-1}$, we have

$$\|I1_{\mathscr{S}}\|^{(1)}_{G^*, \eta^{-1}\Gamma^*, h^*, u^j} \leqslant C_\eta L^{-1} \|K^\# 1_{\mathscr{S}}\|_{G^\#, \Gamma^\#, h^*, u^j} \tag{155}$$

$$\|I_{p>0}1_{\mathscr{S}}\|^{(1)}_{G^*, \eta^{-1}\Gamma^*, h^*, u^j} \leqslant C_\eta L^{d/2-1} \|I_{p>0}1_{\mathscr{S}}\|_{G^\#, \Gamma^\#, h^*, u^j} \tag{156}$$

$$\|I_0 1_{\mathscr{S}}\|^{(1)}_{G^*, \eta^{-1}\Gamma^*, h^*} \leqslant C_\eta L^d \|I_0 1_{\mathscr{S}}\|_{G^\#, \Gamma^\#, h^*} \tag{157}$$

The rest of the argument for small sets relies on the fact that the relevant parts of $K^\# 1_{\mathscr{S}}$ have been extracted. A result of Brydges and Yau (ref. 1, Lemma 4.3) is applicable to functionals whose low-order ψ

derivatives vanish at $\psi = 0$ and we can adapt their proof. Since $I_{p>0} 1_{\mathscr{S}}$ has no constant term in $\partial\phi$, we have

$$\begin{aligned} \|I_{p>0} 1_{\mathscr{S}}\|_{G^{\#}, \Gamma^{\#}, h^{*}, u^{j}} &\leqslant C_{\eta} \|I_{p>0} 1_{\mathscr{S}}\|_{G^{\#}, \Gamma^{\#}, h^{*}, u^{j}, \dim \geqslant d/2} \\ &\leqslant C_{\eta} L^{-d/2} \|I_{p>0} 1_{\mathscr{S}}\|_{G^{\#}, \Gamma^{\#}, h^{\#}, u^{j}, \dim \geqslant d/2} \\ &\leqslant C_{\eta} L^{-d/2} \|(K^{\#}_{p>0}) 1_{\mathscr{S}}\|_{G^{\#}, \Gamma^{\#}, h^{\#}, u^{j}} \end{aligned} \tag{158}$$

[The first C_η actually has the form $C_\eta = 1 + C_s L^{d/2} (\kappa^j h_0^2)^{-1/2} = 1 + \mathcal{O}(1-\eta)$, where C_s is a Sobolev constant.] The factor $L^{-d/2}$ comes from changing h^* to $h^{\#}$ in $\dim \geqslant d/2$ terms. Similarly, we have

$$\begin{aligned} \|I_0 1_{\mathscr{S}}\|_{G^{\#}, \Gamma^{\#}, h^{*}} &\leqslant C_{\eta} \|I_0 1_{\mathscr{S}}\|_{G^{\#}, \Gamma^{\#}, h^{*}, \dim \geqslant d+1} \\ &\leqslant C_{\eta} L^{-d-1} \|I_0 1_{\mathscr{S}}\|_{G^{\#}, \Gamma^{\#}, h^{\#}} \\ &\leqslant C_{\eta} L^{-d-1} \|K^{\#}_0 1_{\mathscr{S}}\|_{G^{\#}, \Gamma^{\#}, h^{\#}} \end{aligned} \tag{159}$$

From the above we have

$$\|I 1_{\mathscr{S}}\|^{(1)}_{G^{*}, \eta^{-1}\Gamma^{*}, h^{*}, u^{j}} \leqslant C_{\eta} L^{-1} \|K^{\#} 1_{\mathscr{S}}\|_{G^{\#}, \Gamma^{\#}, h^{\#}, u^{j}} \tag{160}$$

which combines with (155) to complete the proof. ▮

7. ASYMPTOTICS OF CORRELATION FUNCTIONS

We are now in a position to analyze the generating functionals $Z(\rho) = \langle e^{i(\rho, \phi)} \rangle$ and related physical quantities. These are defined on the torus $\Lambda_0 \equiv \Lambda(N) \equiv (Z/L^N)^d$ and our goal is to obtain bounds which are uniform in N and so hold for any infinite-volume limit. The simplest to analyze are the Green's functions $G(x, y)$ and $G^t(x_1,..., x_p)$ given by (7) and (8). We estimate these by comparison with the free-field values ($z = 0$), which are $G(x, y) = v(x, y)$ and $G^t(x_1,..., x_p) = 0$.

Theorem 2. For $0 < \varepsilon < 1$, $u_0 > 0$, and z sufficiently small there is a constant C [independent of N and $\mathcal{O}(z)$ as $z \to 0$] such that with $\sigma = \sigma^N = \sum_{j=0}^{N-1} \delta\sigma^j$

$$\begin{aligned} &|G(x, y) - (1+\sigma)^{-1} v(x, y)| \\ &\quad \leqslant C |x - y|^{-d+1+\varepsilon} \end{aligned} \tag{161}$$

$$\begin{aligned} &|G^t(x_1,..., x_p)| \\ &\quad \leqslant Cp!\, u_0^{-p} T(x_1,..., x_p)^{-d/2+1} \operatorname{diam}(x_1,..., x_p)^{-d/2+\varepsilon}, \qquad p > 2 \end{aligned} \tag{162}$$

It is helpful to also compare the asymptotic behavior of $G(x, y)$ to that of the infinite-volume inverse Laplacian, which can be done using the following lemma:

Lemma 5. For any $0<\varepsilon<1$ and all $x \in \Lambda(N)$

$$\beta^{-1}v(x, 0) = (K_d)^{-1}\,|x|^{-d+2} + \mathcal{O}(|x|^{-d+1+\varepsilon}), \qquad d \geqslant 3 \tag{163}$$

$$\beta^{-1}v(x, 0) = (2\pi)^{-1}\log[L^N/\max(|x|, 1)] + \mathcal{O}(1), \qquad d=2 \tag{164}$$

uniformly in N. The constant K_d equals $(d-2)$ times the volume of the unit sphere in $\mathbf{R}^d$.

Proof. See the end of this section.

Remarks. (1) The bound (161) agrees with the result of Gawedzki and Kupiainen[6] for the lattice model, but the error term here is roughly one power better. The bound (162) is new and says that there is tree decay on the points $x_1,\ldots, x_p$ with a factor $|x_i - x_j|^{-d/2+1}$ on each line and overall diameter decay. Possibly the exponent on the tree decay can be improved.

(2) In $d>2$, as $N\to\infty$ we have that $v(x, y)$ on Λ_0 converges to the cutoff inverse Laplacian on $\mathbf{R}^d$ (times β). By our estimates, G, G^t, and σ are all bounded in N. If they have limits as $N\to\infty$ then (161) and (162) hold for these limits. For the infinite-volume two-point function this means that for $|x-y|$ large

$$G(x, y) = (1+\sigma)^{-1}\,\beta K_d^{-1}\,|x-y|^{-d+2} + \mathcal{O}(|x-y|^{-d+1+\varepsilon}) \tag{165}$$

(3) In $d=2$, $v(x, y)$ converges to the cutoff infinite-volume inverse Laplacian, but only modulo a divergent constant. The thermodynamic limit of the generating functional $Z(\rho)$ for neutral configurations $\int \rho = 0$ should exist. In this limit we will find for $|x-y|$ large

$$G(x, y) = (1+\sigma)^{-1}\,\beta(2\pi)^{-1}\log|x-y|^{-1} + \mathcal{O}(1) \tag{166}$$

as a distribution on neutral test functions.

Proof of Theorem 2. From (43) we have $\log Z(\rho) = -\frac{1}{2}(\rho, w\rho) + \mathscr{E}(\rho)$. Taking functional derivatives gives, for $p=2$ (in which case $G^T = G$),

$$G(x, y) = w(x, y) + \mathscr{E}_2(x, y) \tag{167}$$

and for $p>2$

$$G^T(x_1,\ldots, x_p) = \mathscr{E}_p(x_1,\ldots, x_p) \tag{168}$$

We have

$$w(x, y) = \sum_{j=0}^{N} L^{-j(d-2)} C^j(L^{-j}x, L^{-j}y) \tag{169}$$

where $C^j = C^j_{\sigma^j}$, and for any μ

$$C^j_\mu(x, y) = \begin{cases} \beta\, |\Lambda_j|^{-1} \sum\limits_{p \in \Lambda_j^* \setminus \{0\}} e^{ip(x-y)} p^{-2} [(e^{p^4} + \mu)^{-1} - (e^{L^4 p^4} + \mu)^{-1}] & \text{if } j < N \\ \beta\, |\Lambda_N|^{-1} \sum\limits_{p \in \Lambda_N^* \setminus \{0\}} e^{ip(x-y)} p^{-2} (e^{p^4} + \mu)^{-1} & \text{if } j = N \end{cases} \tag{170}$$

Also,

$$\mathscr{E}_p(x_1, \ldots, x_p) = \sum_{j=0}^{N} L^{-jp(d/2-1)} \mathscr{E}^j_p(L^{-j}x_1, \ldots, L^{-j}x_p) \tag{171}$$

To prove the theorem, it suffices to prove the bound (162) for $\mathscr{E}_p$, $p \geqslant 2$, and the bound

$$|w(x, y) - (1+\sigma)^{-1}\, v(x, y)| \leqslant C\, |x-y|^{-d+1+\varepsilon} \tag{172}$$

We begin with the latter.

Define C^j_* by putting $\mu = \sigma = \sigma_N$ in (170) and w_* by replacing C^j by C^j_* in (169). For w_* we may collapse the sum and obtain

$$w_*(x, y) = \beta\, |\Lambda_0|^{-1} \sum_{p \in \Lambda_0^* \setminus \{0\}} e^{ip(x-y)} p^{-2} (e^{p^4} + \sigma)^{-1} \tag{173}$$

(i.e., v_N on Λ_0 instead of Λ_N). We compare this with v.

Lemma 6. For some $a > 0$

$$|w_*(x, y) - (1+\sigma)^{-1}\, v(x, y)| \leqslant C e^{-a\,|x-y|} \tag{174}$$

Proof. The left side has the form $|F(x-y)|$, where

$$F(x) = |\Lambda|^{-1} \sum_{p \in \Lambda_0^* \setminus \{0\}} e^{ipx} \tilde{f}(p) \tag{175}$$

with

$$\tilde{f}(p) = \beta\sigma(e^{p^4} - 1)\, p^{-2} e^{-p^4} (e^{p^4} + \sigma)^{-1} (1+\sigma)^{-1} \tag{176}$$

But $\tilde{f}$ is also defined on $\mathbf{R}^d$ and we may consider the inverse Fourier transform $f(x)$. Then we have

$$F(x) = \sum_{n \in \mathbf{Z}^d} f(x + nL^N) \tag{177}$$

since both sides are periodic and have Fourier coefficients $\tilde{f}(p)$. Now $\tilde{f}(p)$ is analytic in a small strip $|\text{Im } p| \leqslant a$ around the real axis and the contour can be deformed to obtain

$$|f(x)| \leqslant Ce^{-a|x|}, \qquad x \in \mathbf{R}^d \tag{178}$$

Therefore also

$$|F(x)| \leqslant Ce^{-a|x|}, \qquad x \in \Lambda \tag{179}$$

which gives the result. ∎

The next result completes the proof of (172).

Lemma 7. For any $\varepsilon > 0$ there is a constant C so that

$$|w(x, y) - w_*(x, y)| \leqslant C\, |x - y|^{-d+1+\varepsilon} \tag{180}$$

Proof. We have

$$w(x, y) - w_*(x, y) = \sum_{j=0}^{N-1} L^{-(d-2)j}[C^j(L^{-j}x, L^{-j}y) - C^j_*(L^{-j}x, L^{-j}y)] \tag{181}$$

As in Lemma 6,

$$|C^j_\mu(x, y)| \leqslant Ce^{-a|x-y|} \tag{182}$$

uniformly in $|\mu| < 1/2$, and a Cauchy bound gives, for $|\mu|, |\mu'| < 1/4$,

$$|C^j_\mu(x, y) - C^j_{\mu'}(x, y)| \leqslant C\, |\mu - \mu'|\, e^{-a|x-y|} \tag{183}$$

Now take $\mu = \sigma_j$, $\mu' = \sigma$:

$$\begin{aligned} &|C^j(L^{-j}x, L^{-j}y) - C^j_*(L^{-j}x, L^{-j}y)|\, L^{-(d-2)j} \\ &\quad \leqslant C\, |\sigma_j - \sigma|\, L^{-(d-2)j}\, |x/L^j - y/L^j|^{-d+1+\varepsilon} \\ &\quad \leqslant C\, |\sigma_j - \sigma|\, L^{(1-\varepsilon)j}\, |x - y|^{-d+1+\varepsilon} \end{aligned} \tag{184}$$

The result now follows: since $|\delta\sigma^j| \leqslant \mathcal{O}(L^{-j})$ and hence $|\sigma^j - \sigma| \leqslant \mathcal{O}(L^{-j})$, the sum of (184) over j is bounded using

$$\sum_{j=0}^{N-1} |\sigma_j - \sigma|\, L^{(1-\varepsilon)j} \leqslant C \tag{185}$$

for a constant C. ∎

Lemma 8. For $p \geqslant 1$,

$$|\mathscr{E}_p(\mathbf{x})| \leqslant Cp!\, u_0^{-p} T(\mathbf{x})^{-d/2+1} \operatorname{diam}(\mathbf{x})^{-d/2+\varepsilon} \tag{186}$$

Proof. From (31) we have for $j < N$ that $\mathscr{E}_p^j(L^{-j}\mathbf{x}) = 0$ if $j < J \equiv \min\{j\colon \delta_{L^{-j}\mathbf{x}} \in \mathscr{S}\}$ and

$$\mathscr{E}_p^j(L^{-j}\mathbf{x}) = \sum_{X:\, X \cup \delta_{L^{-j}\mathbf{x}} \in \mathscr{S}} K_{0,p}^{\#,j}(X, 0; L^{-j}\mathbf{x}) \tag{187}$$

if $j \geqslant J$. For $j \geqslant J$, $G^{\#} = G^{\#,j}$, and $\Gamma^{\#} = \Gamma^{\#,j}$, we have

$$\begin{aligned} &|L^{-jp(d/2-1)} \mathscr{E}_p^j(L^{-j}\mathbf{x})| \\ &\quad\leqslant \sup_X (\Gamma^{\#}(X, L^{-j}\mathbf{x}))^{-1} \|K_{0,p}^{\#,j}\|_{G^{\#},\Gamma^{\#}} L^{-jp(d/2-1)} \\ &\quad\leqslant \sup_X (\Gamma^{\#}(X, L^{-j}\mathbf{x}))^{-1} u_0^{-p} p!\, \|K^{\#,j}\|_{G^{\#},\Gamma^{\#},h^{\#},u^j} \end{aligned} \tag{188}$$

where the sup is over X such that $X \cup \delta_{L^{-j}\mathbf{x}} \in \mathscr{S}$. But $j \geqslant J$ implies $\operatorname{diam}(L^{-j}\mathbf{x}) \leqslant 2^d$ and so, by Lemma 1, for such X,

$$\Gamma^{\#}(X, L^{-j}\mathbf{x})^{-1} \leqslant CT(\mathbf{x})^{-d/2+1} L^{-j(d/2-1)} \tag{189}$$

Also we have

$$\|K^{\#,j}\|_{G^{\#},\Gamma^{\#},h^{\#},u^j} \leqslant \delta^j \leqslant L^{-j(1-\varepsilon)}\delta^0$$

and so, for any $J \leqslant j < N$,

$$|L^{-jp(d/2-1)} \mathscr{E}_p^j(L^{-j}\mathbf{x})| \leqslant Cp!\, u_0^{-p} L^{-j(d/2-\varepsilon)} T(\mathbf{x})^{-d/2+1} \tag{190}$$

The last term $j = N$ in (171) is defined differently, but still satisfies the same bound, as we now demonstrate. $\mathscr{E}^N$ is defined on a single block $\Delta_N = \Delta$ and we can regard it as a local analytic functional (independent of ψ) given by

$$\begin{aligned} \mathscr{E}^N(\Delta, \rho) &= \log[1 + K^{\#,N}(\Delta, \rho)] - [\rho = 0] \\ &= \sum_{r=1}^{\infty} (-1)^r r^{-1} [K^{\#,N}(\Delta, \rho)]^r - [\rho = 0] \end{aligned} \tag{191}$$

Let $\mathscr{E}_+^N$ be the same without the $[\rho=0]$ subtraction; this has the same ρ-derivatives. Then one can establish

$$\begin{aligned}\|\mathscr{E}_+^N\|_{\Gamma^\#,u^N} &\leqslant \sum_{r=1}^{\infty} r^{-1}\,\|(K^{\#,N})^r\|_{\Gamma^\#,u^N}\\ &\leqslant \sum_{r=1}^{\infty} r^{-1}\,\|K^{\#,N}\|_{\Gamma^\#,u^N}^r\\ &\leqslant 2L^{-N(1-\varepsilon)}\delta^0 \end{aligned}\tag{192}$$

Now we have, as in (188)–(190),

$$\begin{aligned}|L^{-Np(d/2-1)}\mathscr{E}_p^N(L^{-N}\mathbf{x})| &\leqslant p!\,u_0^{-p}\Gamma^\#(\Delta, L^{-N}\mathbf{x})^{-1}\,\|\mathscr{E}_+^N\|_{\Gamma^\#,u^N}\\ &\leqslant Cp!\,u_0^{-p}L^{-N(d/2-\varepsilon)}T(\mathbf{x})^{-d/2+1}\end{aligned}\tag{193}$$

Putting (190) and (193) together, we have

$$\begin{aligned}|\mathscr{E}_p(\mathbf{x})| &\leqslant \sum_{j=J}^{N} L^{-jp(d/2-1)}\,|\mathscr{E}_p^j(L^{-j}\mathbf{x})|\\ &\leqslant Cp!\,u_0^{-p}L^{-J(d/2-\varepsilon)}T(\mathbf{x})^{-d/2+1}\end{aligned}\tag{194}$$

But $L^{-J}\leqslant 2^d[\operatorname{diam}(\mathbf{x})]^{-1}$ and this completes the proof of Lemma 8 and Theorem 2. ▮

Corollary 1. In $d\geqslant 3$, $\mathscr{E}(\rho)$ and $\log Z(\rho)$ are analytic and uniformly bounded for $\|\rho\|\leqslant B<u_0$ and all N. In $d=2$, $\mathscr{E}(\rho)$ is analytic and uniformly bounded for $\|\rho\|\leqslant B<u_0$.

Remark. u_0 is arbitrary, but a larger u_0 forces a larger h_0 and hence a smaller $|z|$.

Proof. By Lemma 8, $\mathscr{E}_p(\rho^p)\leqslant Cp!\,(\|\rho\|/u_0)^p$ for $p\geqslant 1$ and $\mathscr{E}_0=0$, hence the bound on $\mathscr{E}(\rho)$. Also, $(\rho, w\rho)$ is always uniformly approximated by $(\rho, v\rho)$. In $d\geqslant 3$ (but not in $d=2$), $\|v\|_\infty<\infty$, hence $(\rho, v\rho)$ is uniformly bounded. ▮

Now we turn to the charge–charge correlation functions defined by (6).

Theorem 3. For any charge q, and $|z|$ sufficiently small (depending on q), there is a constant $C>1$, so that for $|x-y|$ large and $d\geqslant 3$,

$$\begin{aligned}&C^{-1}\beta q^2[(1+\sigma)\,K_d]^{-1}\,|x-y|^{-d+2}\\ &\quad\leqslant g^t(x,y)\\ &\quad\leqslant C\beta q^2[(1+\sigma)\,K_d]^{-1}\,|x-y|^{-d+2}\end{aligned}\tag{195}$$

For $d=2$ we have

$$C^{-1}\,|x-y|^{-\beta q^2/2\pi(1+\sigma)} \leqslant g(x, y) \leqslant C\,|x-y|^{-\beta q^2/2\pi(1+\sigma)} \tag{196}$$

and

$$C^{-1}\,|\Lambda_0|^{-\beta q^2/8\pi(1+\sigma)} \leqslant Z(\rho_x) \leqslant C\,|\Lambda_0|^{-\beta q^2/8\pi(1+\sigma)} \tag{197}$$

Thus $Z(\rho_x) \to 0$ as $|\Lambda_0| \to \infty$ and $g^t(x, y)$ satisfies (196) as well.

Remark. These results are consistent with similar bounds proved for lattice dipole gases by Fröhlich and Spencer.[4]

Proof. We assume $|q| \leqslant 1$. Let $\rho_x = q\delta_x$ and $\rho_y = -q\delta_y$, so $g^t(x, y) = Z(\rho_x + \rho_y) - Z(\rho_x)\,Z(\rho_y)$. Also define

$$A(s) = s \log Z(\rho_x + \rho_y) + (1-s)[\log Z(\rho_x) + \log Z(\rho_y)]$$

and then

$$g^t(x, y) = \int_0^1 d/ds\, e^{A(s)}\, ds = A' \int_0^1 e^{A(s)}\, ds \tag{198}$$

By Corollary 1, in $d \geqslant 3$, $A(s)$ is bounded and so it suffices to prove

$$\begin{aligned}(1+\varepsilon)^{-1}\, &\beta q^2[(1+\sigma)\,K_d]^{-1}\,|x-y|^{-d+2}\\ &\leqslant A'\\ &\leqslant (1+\varepsilon)\,\beta q^2[(1+\sigma)\,K_d]^{-1}\,|x-y|^{-d+2}\end{aligned} \tag{199}$$

for $A' = \log Z(\rho_x + \rho_y) - \log Z(\rho_x) - \log Z(\rho_y)$. There are two contributions to this quantity from the two terms in $\log Z(\rho) = -\frac{1}{2}(\rho, w\rho) + \mathscr{E}(\rho)$. The first is just $(\rho_x, w\rho_y) = -q^2 w(x, y)$, and the bound

$$\begin{aligned}(1+\varepsilon)^{-1}\, &\beta[(1+\sigma)\,K_d]^{-1}\,|x-y|^{-d+2}\\ &\leqslant w(x, y)\\ &\leqslant (1+\varepsilon)\,\beta[(1+\sigma)\,K_d]^{-1}\,|x-y|^{-d+2}\end{aligned} \tag{200}$$

follows from Lemmas 5–7.

The second term is $\mathscr{E}(\rho_x + \rho_y) - \mathscr{E}(\rho_x) - \mathscr{E}(\rho_y)$. Expanding each term by $\mathscr{E}(\rho) = \sum_{p=1}^{\infty} (1/p!)\, \mathscr{E}_p(\rho^p)$ gives

$$\mathscr{E}(\rho_x + \rho_y) - \mathscr{E}(\rho_x) - \mathscr{E}(\rho_y) = \sum_{a,b=1}^{\infty} (1/a!\, b!)\, \mathscr{E}_{a+b}(\rho_x^a \rho_y^b) \tag{201}$$

By Lemma 8,

$$|\mathscr{E}_{a+b}(\rho_x^a \rho_y^b)| \leqslant C(a+b)!\, u_0^{-(a+b)}\, |x-y|^{-d+1+\varepsilon} \tag{202}$$

Assuming $u_0 > 2$,

$$\sum_{a,b} (a+b)!/a!\, b!\, u_0^{-(a+b)} = \sum_p (2/u_0)^p = \mathcal{O}(1) \tag{203}$$

and so we conclude that in $d \geqslant 3$

$$|\mathscr{E}(\rho_x + \rho_y) - \mathscr{E}(\rho_x) - \mathscr{E}(\rho_y)| \leqslant C\, |x-y|^{-d+1+\varepsilon} \tag{204}$$

In $d=2$, $A(s)$ and $A'(s)$ are not uniformly bounded, since $\int \rho_x = -\int \rho_y \neq 0$, and so the argument above fails. However, working directly, one finds that

$$Z(\rho_x) = \langle e^{iq\phi(x)} \rangle = \exp\left[-q^2/2 \sum_{j=0}^{N} C^j(0,0) + \mathscr{E}(\rho_x) \right] \tag{205}$$

where $\mathscr{E}(\rho_x)$ is bounded and, as proved in ref. 2, Lemma A.3,

$$|C^j(0,0) - \beta \log L/2\pi(1+\sigma)| \leqslant \mathcal{O}(L^{j-N}) \tag{206}$$

Thus, $Z(\rho_x) \sim |\Lambda_0|^{-\beta q^2/8\pi(1+\sigma)}$, which goes to zero as $|\Lambda_0| \to \infty$.

For

$$g(x,y) = Z(\rho_x + \rho_y) = \exp[-q^2(w(0,0) - w(x,y)) + \mathscr{E}(\rho_x + \rho_y)]$$

we note that $\mathscr{E}(\rho_x + \rho_y)$ is bounded by Corollary 1, and $|w(0,0) - w(x,y)|$ has a $\log |x-y|$ bound by Lemmas 5–7. Thus we conclude that in $d=2$

$$g(x,y) \leqslant C \exp[-\beta q^2 \log |x-y|/2\pi(1+\sigma)] = C\, |x-y|^{-\beta q^2/2\pi(1+\sigma)} \tag{207}$$

The corresponding lower bound follows in the same way. ∎

Finally, there remains the following task.

Proof of Lemma 5. We sketch the proof of the harder case $d=2$, with the momentum cutoff function e^{-p^2} rather than e^{-p^4}. First we compare

$$v(x,0) = \sum_{j=0}^{N} C_0^j(L^{-j}x, 0) \tag{208}$$

to the infinite-volume function

$$v_{\infty}(x, 0) = \beta(2\pi)^{-2} \int dp\, e^{ipx} p^{-2}(e^{-p^2} - e^{-L^{2N}p^2})$$

$$= \sum_{j=0}^{N-1} C_{0,\infty}(L^{-j}x, 0) \tag{209}$$

where

$$C_{0,\infty} = \beta(2\pi)^{-2} \int dp\, e^{ipx} p^{-2}(e^{-p^2} - e^{-L^2p^2})$$

As in the proof of Lemma 6, for $j < N$,

$$C_0^j(L^{-j}x, 0) = \sum_{n \in \mathbf{Z}^d} C_{0,\infty}(L^{-j}x + nL^{N-j}, 0) \tag{210}$$

Since $C_{0,\infty}(x, 0) \leqslant \mathcal{O}(1)\, e^{-a|x|}$ for some $a = \mathcal{O}(1)$,

$$|C_0^j(L^{-j}x, 0) - C_{0,\infty}(L^{-j}x, 0)| \leqslant \sum_{\substack{n \in \mathbf{Z}^d \\ n \neq 0}} \mathcal{O}(1)\, e^{-a|L^{-j}x + nL^{N-j}|}$$

$$\leqslant \mathcal{O}(1)\, e^{-aL^{N-j}/2} \tag{211}$$

and

$$\sum_{j=0}^{N-1} |C_0^j(L^{-j}x, 0) - C_{0,\infty}(L^{-j}x, 0)| \leqslant \mathcal{O}(1) \sum_{j=0}^{N-1} e^{-aL^{N-j}/2}$$

$$\leqslant \mathcal{O}(1) \tag{212}$$

One can also show $|C_0^N(L^{-N}x, 0)| \leqslant \mathcal{O}(1)$.

Now,

$$\beta^{-1} v_{\infty}(x, 0) = (2\pi)^{-2} \int dp\, e^{ipx} p^{-2} \int_1^{L^{2N}} ds\, (-d/ds)\, e^{-sp^2}$$

$$= (4\pi)^{-1} \int_1^{L^{2N}} ds\, s^{-1} e^{-x^2/4s} \tag{213}$$

For $1 \leqslant x^2 \leqslant L^{2N}$, write

$$\beta^{-1} v_{\infty}(x, 0) = (4\pi)^{-1} \int_{x^2}^{L^{2N}} ds\, s^{-1} + (4\pi)^{-1} \int_1^{x^2} ds\, s^{-1} e^{-x^2/4s}$$

$$+ (4\pi)^{-1} \int_{x^2}^{L^{2N}} ds\, s^{-1}(e^{-x^2/4s} - 1) \tag{214}$$

The first term gives $(2\pi)^{-1}\log(L^N/|x|)$, while the second and third terms can be shown to be $\mathcal{O}(1)$. For $x^2<1$ one writes

$$\beta^{-1}v_\infty(x,0)=(4\pi)^{-1}\int_1^{L^{2N}} ds\, s^{-1}+(4\pi)^{-1}\int_1^{L^{2N}} ds\, s^{-1}(e^{-x^2/4s}-1) \qquad (215)$$

and shows that the second term is $\mathcal{O}(1)$, to complete the proof. ∎

APPENDIX

Let $G_\kappa(X)$ be given by (47).

Lemma 9. Let $\{X_i\}$ be a set of polymers, and $X=\bigcup_i X_i$. Suppose the maximum overlap $\tau=\sup_{x\in\Lambda}\#\{i: X_i\ni x\}$ is finite. Then

$$G_\kappa(X)\geqslant\prod_i G_{\kappa/\tau}(X_i) \qquad (A1)$$

Proof. For any polymer X, let $\tilde{X}\supset X$ denote any extension of X obtained by replacing some or all faces f of ∂X by $f\cup f_{1/2}$, where $f_{1/2}$ is the open rectangle of width 1/2 satisfying $\bar{f}_{1/2}\cap X=f$. We note that two polymers X, Y overlap if and only if $\tilde{X}, \tilde{Y}$ overlap. Therefore, $\tau=\sup_{x\in\Lambda}\#\{i: \tilde{X}_i\ni x\}$. Also, by the Sobolev inequality, $\|\partial\phi\|_f^2\leqslant C_s\,\|\partial\phi\|_{s,f_{1/2}}^2$.

To prove (A1), we define $\tilde{X}_i$ by extending faces $f\in\partial X_i\backslash\partial X$. Since we only extend interior faces, we still have $X=\bigcup_i\tilde{X}_i$ and hence

$$\sum_i(\|\partial\phi\|_{s,X_i}^2+1/c\,\|\partial\phi\|_{\partial X_i}^2)\leqslant\sum_i\left(\|\partial\phi\|_{s,\tilde{X}_i}^2+1/c\sum_{f\in\partial X_i\cup\partial X}\|\partial\phi\|_f^2\right)$$
$$\leqslant\tau(\|\partial\phi\|_{s,X}^2+1/c\,\|\partial\phi\|_{\partial X}^2)$$

where for the first inequality we require $C_s d/c\leqslant 1$. ∎

Now we consider the one-parameter family of large-field regulators $g(t,X,\phi)=g_\kappa(t,X,\phi)$, $0\leqslant t\leqslant 1$, defined by (144), and the Gaussian measure $d\mu_C$ on Λ with covariance C given by (21). The following result extends ref. 1, Proposition 9.1 to our situation.

Proposition 6. There exists $\kappa_{\max}>0$ such that for all $\kappa\leqslant\kappa_{\max}$ and $0\leqslant u\leqslant t\leqslant 1$,

$$\mu_{(t-u)C}*g(u)\leqslant g(t) \qquad (A2)$$

Proof. For simplicity, we treat the large-field regulators as though they depend only on $\partial\phi$, $\partial^2\phi$:

$$g(u,X,\phi)=e^{\delta u\,|X|}E(X,\phi)\,F_1(u,X,\phi)\,F_2(u,X,\phi) \qquad (A3)$$

where $\delta = \log \gamma$,

$$E(X, \phi) = \exp\left(\tfrac{1}{2}\kappa \int_X [\partial\phi]^2\right) \tag{A4}$$

$$F_1(X, \phi) = \exp\left(\tfrac{1}{2}\kappa c^{-1} l_1(u) \int_{\partial X} [\partial\phi]^2\right) \tag{A5}$$

$$F_2(X, \phi) = \exp\left(\tfrac{1}{2}\kappa l_2(u) \int_X [\partial^2\phi]^2\right) \tag{A6}$$

$$l_i(u) = L^{-i} + (1 - L^{-i})u \tag{A7}$$

By Holder's inequality,

$$\mu_C * g(u) \leqslant e^{\delta u\,|X|}(\mu_c * E^3)^{1/3}\,(\mu_c * F_1^3)^{1/3}\,(\mu_c * F_2^3)^{1/3} \tag{A8}$$

For any Gaussian function $G(\phi) = e^{\beta\langle S\phi, \phi\rangle/2}$, where $S: \mathscr{H}_s(\Lambda) \to \mathscr{H}_{-s}(\Lambda)$ is such that CS is a trace-class self-adjoint operator on $\mathscr{H}_s(\Lambda)$, we have

$$\begin{aligned}[\mu_{\alpha C} * G](\phi) &= G(\phi) \int d\mu_{\alpha C}(\zeta) \exp \beta[\langle S\zeta, \zeta\rangle/2 + \langle S\phi, \zeta\rangle]\\ &= G(\phi) \exp[\alpha\beta^2 \langle S\phi, (1 - \alpha\beta CS)^{-1}\, CS\phi\rangle]\\ &\quad \times \det{}_{L_2}^{-1/2}(1 - \alpha\beta C^{1/2} S C^{1/2})\end{aligned}$$

To get $\mu_c * F_1^3$, we take CS_1 given by

$$(CS_1\phi)(x) = \int_{\partial X} (\partial C)(x, y)(\partial\phi)(y)\, dy \tag{A9}$$

Provided $\|CS_1\|_{\mathscr{H}_s} \leqslant \mathcal{O}(1)$ and $\mathrm{tr}_{L_2} C^{1/2} S_1 C^{1/2} \leqslant \mathcal{O}(1)\,|X|$, and κ is chosen small, we find that for $\beta = 3\kappa l_1(u)/c$, $\alpha = t - u$,

$$\begin{aligned}\det{}_{L_2}^{-1/2}(1 - \alpha\beta C^{1/2} S_1 C^{1/2}) &\leqslant \exp[(\alpha\beta/2)(\mathrm{tr}\, C^{1/2} S_1 C^{1/2})(1 - \alpha\beta\, \|CS_1\|)^{-1}]\\ &\leqslant \exp[\mathcal{O}(1)\,\kappa(t - u)\,|X|]\end{aligned}$$

and

$$\begin{aligned}&\exp[\alpha\beta^2 \langle S_1\phi, (1 - \alpha\beta CS_1)^{-1}\, CS_1\phi\rangle]\\ &\quad \leqslant \exp[\mathcal{O}(1)\,\kappa^2(t - u)\,\|\partial\phi\|_{\partial X}^2]\end{aligned}$$

Thus

$$\begin{aligned}[\mu_{(t-u)C} * F_1^3(u)]^{1/3} &\leqslant F_1(u, X, \phi) \exp[\mathcal{O}(1)\,\kappa(t - u)|X|]\\ &\quad \times \exp[\mathcal{O}(1)\,\kappa^2(t - u)\,\|\partial\phi\|_{\partial X}^2]\end{aligned}$$

A similar argument shows

$$[\mu_{(t-u)C} * F_2^3(u)]^{1/3} \leqslant F_2(u, X, \phi) \exp[\mathcal{O}(1)\,\kappa(t-u)\,|X|] \times \exp[\mathcal{O}(1)\,\kappa^2(t-u)\,\|\partial\phi\|_X^2]$$

provided CS_2 defined by

$$(CS_2\phi)(x) = \int_X (\partial^2 C)(x, y)(\partial^2\phi)(y)\,dy \tag{A10}$$

has $\|CS_2\|_{\mathscr{H}_s} \leqslant \mathcal{O}(1)$ and $\mathrm{tr}_{L_2}\, C^{1/2} S_2 C^{1/2} \leqslant \mathcal{O}(1)\,|X|$, and κ is chosen small enough.

The dangerous factor E is treated using integration by parts:

$$\begin{aligned}
&[\mu_{(t-u)C} * E^3(u)]^{1/3} \\
&\quad = E(\phi)\left\{\int d\mu_{(t-u)C}(\zeta) \right.\\
&\qquad \left.\times \exp(3\kappa/2)\left(\int_X [\partial\zeta]^2 + 2\int_{\partial X} [\zeta\,\partial\phi] - 2\int_X [\zeta\,\partial^2\phi]\right)\right\}^{1/3} \\
&\quad \leqslant E(u, X, \phi)\exp[\mathcal{O}(1)\,\kappa(t-u)\,|X|] \\
&\qquad \times \exp[\mathcal{O}(1)\,\kappa^2(t-u)(\|\partial\phi\|_{\partial X}^2 + \|\partial\phi\|_{\partial X}^2)]
\end{aligned}$$

The bound above follows, provided

$$(CS_3\phi)(x) = \int_{\partial X} C(x, y)\,\partial\phi(y)\,dy \tag{A11}$$

and

$$(CS_4\phi)(x) = \int_X C(x, y)\,\partial^2\phi(y)\,dy \tag{A12}$$

are such that $\|CS_i\| \leqslant \mathcal{O}(1)$ and $\mathrm{tr}\, C^{1/2} S_i C^{1/2} \leqslant \mathcal{O}(1)\,|X|$. We put all this together using $l_i(u) = l_i(t) - (1 - L^{-i})(t-u)$,

$$\begin{aligned}
[\mu_{(t-u)C} * g(u)] &\leqslant g(t)(\exp\{[-\delta + \mathcal{O}(1)\kappa](t-u)\,|X|\} \\
&\quad \times \exp\{[-\tfrac{1}{2}\kappa(1 - L^{-2}) + \mathcal{O}(1)\,\kappa^2](t-u)\,\|\partial^2\phi\|_X^2\} \\
&\quad \times \exp\{[-\tfrac{1}{2}\kappa(1 - L^{-1}) + \mathcal{O}(1)\,\kappa^2](t-u)\,\|\partial\phi\|_{\partial X}^2\}) \\
&\leqslant g(t)
\end{aligned}$$

provided κ is small. Finally, we note that the required bounds on CS_i follow from bounds for $\partial^\alpha C$ similar to (182). ▮

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